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The Polynomial Method over Varieties

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Abstract

In 2010, Guth and Katz introduced the polynomial partitioning theorem as a tool in incidence geometry and in additive combinatorics. This allowed the application of results from algebraic geometry (mainly on intersection theory and on the topology of real algebraic varieties) to the solution of long standing problems, including the celebrated Erdős distinct distances problem. Recently, Walsh has extended the polynomial partitioning method to an arbitrary subvariety. This result opens the way to the application of this method to control the point-hypersurface incidences and, more generally, of variety-variety incidences, in spaces of arbitrary dimension. This final project consists in studying Walsh's paper, to explain its contents and explore its applications to this kind of incidence problems.

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This work is dedicated to the loving memory of my grandfather.

Chapter 1

Introduction

The polynomial method is a collection of results which allow us to find a polynomial of bounded degree that is able to capture the underlying algebraic structure of a problem involving a finite set of points.
The two most well-known tools of the polynomial method are *Siegel's lemma* and the *polynomial partitioning theorem* over $\mathbb{R}^n$. The former was used in 2008 by Dvir [Dvi08] to prove the finite field Kakeya conjecture while the latter was invented and used by Guth and Katz in 2010 [GK15] to solve a long-standing problem of Erdős [Erd70] about distances in $\mathbb{R}^3$. The polynomial method has also been used to establish results in number theory and differential geometry [Gut16, Tao14, Dvi10].

The current tools of the polynomial method are based on techniques from algebraic geometry (intersection theory), commutative algebra (Hilbert functions), general topology (ham-sandwich theorem), topology of real algebraic varieties and real algebraic geometry.

The general theory of the polynomial method is still being developed. There is a lack of understanding of its limitations and it is thought that there is a lot of room for improvement. One direction that has already been explored is the generalisation of the main results of the polynomial method to algebraic varieties of arbitrary dimension. This was done by Walsh in 2018 [Wal18] and the detailed study of his recent contributions to the polynomial method is the core of this work.

More precisely, in chapter 4 we will study a generalisation of Siegel's lemma and in chapter 5 we will prove the polynomial partitioning theorem for irreducible algebraic varities of arbitrary dimension. We will also be interested in finding an upper bound for the number of connected components of the complement of the zero set of a polynomial that can be intersected by the real points of a variety. This will be done in chapter 6. Using the last two results, in chapter 7 we will be able to prove a general estimate on the number of incidences between points lying in an algebraic variety and a finite set of hypersurfaces. Finally, for reasons that will become clear later, in chapter 8 we will be interested in bounding the number of connected components of an algebraic set.

We continue this introduction by presenting the main ideas which are at the core of the polynomial method and by explaining the motivation to obtain general results for algebraic varieties.

The basic idea to capture a set of objects in the zero set of a polynomial (the points where the polynomial vanishes) received a large amount of attention when Dvir [Dvi08] was able to give a one page proof of the finite field Kakeya conjecture.

Definition 1.0.1 (Finite field Kakeya set)**.** A set $\mathcal{K} \subset \mathbb{F}_q^n$ is a *Kakeya set* if for every $a \in \mathbb{F}_q^n \setminus \{0\}$ there exists a point $b \in \mathbb{F}_q^n$ such that the line $\{b + at \mid t \in \mathbb{F}_q\}$ is contained in $\mathcal{K}$.

Notation 1.0.2. Given any set of objects $\mathcal{S}$, we will denote by $|\mathcal{S}|$ the cardinality of $\mathcal{S}$.

Theorem 1.0.3. *Let $\mathcal{K} \subset \mathbb{F}_q^n$ be a Kakeya set. Then*

$$|\mathcal{K}| \geq (10n)^{-n} \cdot q^n.$$

Although of interest in its own right, the above theorem is a toy model introduced by Wolff as a conjecture [Wol96] in an attempt to make progress on the Kakeya problem over $\mathbb{R}^n$. This problem was mooted at the beginning of the 20th century and despite of a lot of effort, it remains unsolved at the time of this writing. The interest in this question comes from its connection to other famous open problems in the field of harmonic analysis. For example, the restriction problem has been shown to imply the Kakeya conjecture over $\mathbb{R}^n$ [Wol96].

Definition 1.0.4 (Kakeya set over $\mathbb{R}^n$)**.** A *Kakeya set* $\mathcal{K} \subset \mathbb{R}^n$ is a compact set containing a line segment of unit length in every direction.

Conjecture 1.0.5. *Let $\mathcal{K} \subset \mathbb{R}^n$ be a Kakeya set. Then* $\dim(K) = n$*, where* $\dim(K)$ *stands for the Hausdorff dimension of $\mathcal{K}$.*

The Kakeya conjecture over $\mathbb{R}^n$ is trivial for the one dimensional case. The two dimensional case was proven in 1971 by Davis [Dav71, Theorem 2]. The research for $n \geq 2$ started with the work of Bourgain [Bou91] in 1991 and despite of a lot of partial progress (see [KT02] for example) it remains open to this day. The difficulty on tackling this problem was the reason for which the strikingly simple proof of the finite field Kakeya conjecture by Dvir was considered a breakthrough. It is important to remark that no other proof of the finite field Kakeya problem has been found after Dvir's work.

The following lemma is what is considered to be the first basic tool of the polynomial method and is the key technique used in Dvir's proof of the finite field Kakeya conjecture. We will see the proofs of Theorem 1.0.3 and Lemma 1.0.7 in chapter 2.

Notation 1.0.6. Given a parameter n we will write $X \lesssim_n Y$ when there exists a constant C depending only on n such that $X \leq CY$.

Lemma 1.0.7 (Siegel's lemma)**.** *Let $\mathbb{F}$ be a field. For every finite set $\mathcal{S} \subseteq \mathbb{F}^n$ there exists a nonzero polynomial $P \in \mathbb{F}[x_1, \ldots, x_n]$, of degree $\lesssim_n |\mathcal{S}|^{1/n}$, vanishing on $\mathcal{S}$.*

Since the polynomial method is able to identify the underlying algebraic structure of a problem, once a result is proven using this set of techniques it makes sense to try to apply it to closely related questions. The field of discrete geometry has benefited enormously from the use of the polynomial method and it is a clear example of this. To give a concrete example, after Dvir's proof of Theorem 2.1.11, the polynomial method was successfully used by Guth and Katz [GK10] to prove the *joints problem over* $\mathbb{R}^3$, which was first put forward in the 1990s [CEG+91].

A *joint* is defined to be a point where three lines with linearly independent directions intersect. We are interested in knowing how many joints a set of lines determines in the plane.

Theorem 1.0.8 (Guth-Katz [GK10])**.** *Let $\mathcal{L}$ be a set of L lines in $\mathbb{R}^3$ and let $\mathcal{J}$ be the set of J joints determined by $\mathcal{L}$. Then*

$$J \leq 10L^{3/2}.$$

We will see a proof of the above theorem in chapter 2.

Another important problem which was solved using the polynomial method was the *distinct distances problem*, posed by Paul Erdős [Erd70] in 1946 and which is considered to be the starting point of the subfield of discrete geometry called *incidence geometry*, which will be discussed in more detail afterwards.

The distinct distances problem asks how few distinct distances are determined by N points in the plane. More formally (following the notation of [GK15]) we let $P \subset \mathbb{R}^2$ be a set of N points and $d(P)$ be the set of nonzero distances among points of P:

$$d(P) := \{d(p, q)\}_{p,q \in P, p \neq q}.$$

The distinct distances problem asks how small can the set $d(P)$ be. Erdős conjectured that for any configuration of N points in the plane, the number of distinct distances is

$$\gtrsim \frac{N}{\sqrt{\log N}}.$$

Upon trying to adapt the techniques of Dvir's proof to the disctinc distances problem, Guth and Katz [GK15] discovered what is now considered another important tool of the polynomial method, the *polynomial partitioning theorem.*

Notation 1.0.9. Given polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$ we define their *zero set* as

$$Z(f_1, \ldots, f_r) := \{x \in \mathbb{C}^n \ : \ f_1(x) = \cdots = f_r(x) = 0\}.$$

Theorem 1.0.10 (Polynomial partitioning of $\mathbb{R}^n$, Guth-Katz [GK15]). *For every finite set $\mathcal{S} \subset \mathbb{R}^n$ and every choice of an integer $M \geq 1$, there exists some nonzero polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $\lesssim_n M$ such that each connected component of $\mathbb{R}^n \setminus Z(P)$ contains at most*

$$\lesssim_n \frac{|\mathcal{S}|}{M^n}$$

points of $\mathcal{S}$.

Based on ideas of Elekes and Sharir [ES11] which reduce the distinct distances problem to a problem of incidences in space, Guth and Katz were able to use Theorem 1.0.10 to give a proof of the following result.

Theorem 1.0.11 (Guth-Katz [GK15]). *A set of N points in the plane determines*

$$\gtrsim \frac{N}{\log N}$$

distinct distances.

This was considered a breakthrough and drew a lot of attention to the polynomial method again. For example, the polynomial partitioning theorem was used to make improvements to restriction estimates in Fourier analysis by Guth in [Gut14, Gut18] and by Ou and Wang in [OW17] motivated by the work of Du, Guth and Li [DGL17] which also used the partitioning technique. In the field of incidence geometry, it was used by Wang et al. in [WYZ13] to bound the number of incidences between points and (higher degree) algebraic curves. This list is far from exhaustive and we will expand on it later on, when further discussing the polynomial partitioning theorem and the problems that it arises.

Due to its importance for the polynomial method, we will give an example of how to use Theorem 1.0.10. For this we will talk about incidence geometry in more detail.

Incidence geometry is interested in answering questions regarding how simple geometric objects (such as lines, circles, points...) intersect each other. More generally, let V be an algebraic variety over a field $\mathbb{F}$ and let $\mathcal{T}$ be a finite family of subvarieties of V. Let $\mathcal{S}$ be a finite set of points inside of V. Incidence geometry is concerned with how the *number of incidences* between $\mathcal{S}$ and $\mathcal{T}$,

$$I(\mathcal{S}, \mathcal{T}) = |\{(\mathcal{S}, \mathcal{T}) \in S \times T \ : \ s \in t\}|$$

varies in relation to the sizes of $\mathcal{S}$ and $\mathcal{T}$. Taking $V = \mathbb{R}^2$ and $\mathcal{T}$ to be an arbitrary finite family of lines, we may be interested in finding a bound for the maximal number of incidences that can occur between a set of points and a set of lines. The classical result of Szemeŕedi and Trotter [ST83] gives the bound:

$$I(\mathcal{S}, L) \lesssim |\mathcal{S}|^{2/3}|L|^{2/3} + |\mathcal{S}| + |L|.$$

The strategy of proving the Szemerédi-Trotter theorem using the polynomial method is the following. Let $\mathcal{S} \subset \mathbb{R}^2$ be a finite set of points and let $\mathcal{L} \subset \mathbb{R}^2$ be a finite set of lines. The polynomial partitioning theorem tells us that there exists a polynomial P of bounded degree such that each connected component of $\mathbb{R}^2 \setminus Z(P)$ contains few points of $\mathcal{S}$. We now need to know the following basic fact: a line in the plane can only cut at most $\leq D+1$ of the connected components of $\mathbb{R}^2 \setminus Z(P)$, where $P \in \mathbb{R}[x,y]$ is a polynomial of degree at most D. We can then take a divide and conquer approach. Bound the number of incidences within each connected component of $\mathbb{R}^2 \setminus Z(P)$ and then take the sum of these weaker bounds. If the partition was good enough we expect to obtain a better global estimate for the total number of incidences between $\mathcal{S}$ and $\mathcal{L}$ than the one provided by the weaker bound. We will see the complete proof in chapter 2.

Although the polynomial partitioning theorem is a powerful result it has a major caveat. It can happen that a lot of points lie inside of the zero set of the polynomial that it produces. We are then forced to study what occurs inside the variety defined by such polynomial.
A way to proceed is to extend the polynomial partitioning theorem to varieties of lower dimension. This would allow us to iteratively partition the points within the zero set until a suitable bound is obtained. This idea lead to a large body of work where the partitioning technique was extended to handle particular situations, see for example [ST12, Gut15], [Zah15, Theorem 2.3], [FPS+14, Theorem 4.2], [MP15, Theorem 1.1] and [BS16].
It is important to highlight the work of Barone and Sombra [BS16] since their ideas were key to the the extensions of the polynomial method done by Walsh in [Wal18]. Barone and Sombra were able to obtained a partitioning theorem for points in an irreducible algebraic variety of codimension at most two and formulated a series of conjectures that would greatly improve the partitioning technique if shown to be true. In [Wal18], Walsh was able to give an affirmative answer the these conjectures. More specifically, Theorem 1.4 (Theorem 1.0.19) and Theorem 1.5 (Theorem 1.0.15) of [Wal18] prove Conjecture 2.10 of [BS16] and Theorem 1.1 (Theorem 1.0.14) and Corollary 1.7 (Corollary 7.0.5) of [Wal18] solve Conjecture 3.4 and Conjecture 4.1 of [BS16] respectively.
One of the key ideas that these two papers want to convey is that it is possible to obtain a polynomial partitioning theorem over algebraic varieties of arbitrary dimension that produces better bounds as the degree of the varieties grows.

In the remaining of this introduction we will concentrate on giving an overview of the results introduced in [Wal18]. To state them in full generality we would need a collection of definitions which will be discussed in section chapter 3. For ease of exposition we present here less general statements (also given in [Wal18]) which require as little notation as possible.

Notation 1.0.12. Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety. We will let

$$\delta(V)$$

stand for the minimal integer such that V is an irreducible component of $Z(f_1, \ldots, f_r)$

for some polynomials f_i of degree at most $\delta(V)$.

In chapter 4 we will study the generalisation of Siegel's lemma, Lemma 1.0.7. In its simplified form, the result says the following.

Theorem 1.0.13. *Let $0 \leq l < d \leq n$ be integers. Let $V \subseteq \mathbb{C}^n$ be a d-dimensional algebraic set in $\mathbb{C}^n$ and $\tau_l > 0$ a real number. Let T be an l-dimensional algebraic set of $\mathbb{C}^n$ with $\deg(\mathcal{T}) \geq \tau_l \delta(V)^{d-l} \deg(V)$. Then, there exists some polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ of degree at most*

$$\lesssim_{n,\tau_l} \left(\frac{\deg(\mathcal{T})}{\deg(V)}\right)^{\frac{1}{d-l}},$$

vanishing at all elements of T without vanishing identically on V.

As we mentioned before, it is clear that this bound improves as the degree of V gets larger. Furthermore, there is no restriction imposed on the degree of V, only on the degree of T. The general result is Theorem 4.0.3.

Let us now continue with the generalisation of the polynomial partitioning theorem, Theorem 1.0.10.

Theorem 1.0.14. *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d and $\mathcal{S}$ a finite set of points inside of $V(\mathbb{R})$. Then, given any integer $M \geq \delta(V)$, we can find some polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $O_n(M)$ such that P does not vanish identically on V and each connected component of $\mathbb{R}^n \setminus Z(P)$ contains*

$$\lesssim_n \frac{|\mathcal{S}|}{M^d \deg(V)}$$

elements of $\mathcal{S}$.

As with Theorem 1.0.13, the dependence on the degree of V is made explicit and clearly shows that the quantity of points of $\mathcal{S}$ in each connected component (also called cell) of $\mathbb{R}^n \setminus Z(P)$ decreases as the degree of V increases. The general version, Theorem 5.2.2, allows us to freely choose $M \geq 1$.

We have seen that the application of the polynomial partitioning theorem to the proof of the Szemerédi–Trotter theorem relies on the fact that few lines can touch few of the connected components in the complement of the zero set of a polynomial. In order to be able to use Theorem 5.2.2 we will be interested in obtaining control on the higher-dimensional components of the zero set of a tuple of polynomials. To this end, we will use the concept of an *envelope* of an algebraic variety. In chapter 6, after establishing some estimates regarding envelopes we will be able to prove the following theorem.

Theorem 1.0.15. *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d and $P \in \mathbb{R}[x_1, \ldots, x_n]$. Then $V(\mathbb{R})$ intersects $\lesssim_n \deg(V) \deg(P)^d$ connected components of $\mathbb{R}^n \setminus Z(P)$.*

With Theorem 5.2.2 and Theorem 1.0.15 we will be able to show the following general version of the Semerédi-Trotter theorem.

Definition 1.0.16. Let $\mathcal{S}$ be a finite set of points in $\mathbb{R}^n$ and $\mathcal{T}$ a finite set of varieties in $\mathbb{R}^n$. We say $\mathcal{S}$ is (k,b)-free with respect to $\mathcal{T}$ if, for every choice of k distinct elements $s_1,\dots,s_k$ from $\mathcal{S}$ and b distinct elements $t_1,\dots,t_b$ from $\mathcal{T}$, we have $\mathcal{S}_i \notin t_j$ for some $1 \le i \le k$, $1 \le j \le b$.

Notation 1.0.17. We will use the following quantities.

$$\alpha_k(d) = \frac{k(d-1)}{dk-1}, \quad \beta_k(d) = \frac{d(k-1)}{dk-1} \quad \text{and} \quad \tau_d(b,k) = b^{1-\beta_k(d)}k^{1-\alpha_k(d)}.$$

We set $\alpha_1(1) = 0$ and $\beta_1(1) = 1$.

Theorem 1.0.18. *Let $V \subseteq \mathbb{C}^n$ be an irreducible variety of dimension d. Let $\mathcal{T}$ be a set of hypersurfaces of $\mathbb{C}^n$ and $\mathcal{S} \subseteq V(\mathbb{R})$ a set of points that is (k,b)-free with respect to $\mathcal{T}$. Then $I(\mathcal{S},\mathcal{T})$ is bounded by*

$$c_1|\mathcal{S}|^{\alpha_k(d)} \deg(\mathcal{T})^{\beta_k(d)} \deg(V)^{1-\alpha_k(d)} + k\deg(\mathcal{T})\deg(V) + (b-1)|\mathcal{S}|$$

with $c_1 \lesssim_n \tau_d(b,k)$.

We finish the introduction by discussing the 0th Betti number (the number of connected components) of a real algebraic variety $V \subseteq \mathbb{R}^n$. This quantity is very important for the application of the polynomial partitioning theorem. Remember that Theorem 1.0.10 tells us that, for a finite set of points $\mathcal{S} \subset \mathbb{R}^n$ there exists some polynomial P of degree at most D such that each connected component of $\mathbb{R}^n \setminus Z(P)$ contains less than $|\mathcal{S}|/D^n$ points of $\mathcal{S}$. By the seminal work of Milnor [Mil64] and Thom [Tho65] in the 1960's we know that the number $b_0(V)$ of connected components of V is at most

$$\lesssim_n D^n$$

if V is defined by polynomials of degree at most D. This is the reason for which we can suppose that if $\mathcal{S} \subset \mathbb{R}^2 \setminus Z(P)$, then the points must be (almost) equidistributed among the connected components of $\mathbb{R}^2 \setminus Z(P)$. We would like to have a similar estimate for general algebraic varieties. In particular, looking at the bound given by Theorem 1.0.14 we would like to find that the number of connected components of V is at most $\deg(V)\delta(V)^d$. Fortunatly the following theorem says that this is indeed the case.

Theorem 1.0.19. *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d. Then there exists some algebraic set $X \subseteq \mathbb{C}^n$, having V as an irreducible component, with $\deg(X) \lesssim_n \deg(V)$ and such that the number $b_0(X(\mathbb{R}))$ of connected components of $X(\mathbb{R})$ satisfies*

$$b_0(X(\mathbb{R})) \lesssim_n \delta(V)^d \deg(V).$$

We will see the proof of the above theorem in chapter 8.

Chapter 2

Simple examples of the polynomial method

The aim of this chapter is twofold. First, we will prove *Siegel's lemma* (Corollary 2.1.3) which is the most basic tool of the polynomial method. We will see how to use this result to give short and simple proofs of the Nikodym (Theorem 2.1.8) and Kakeya (Theorem 2.1.11) conjectures over finite fields and of the joints problem over $\mathbb{R}^3$ (Theorem 2.1.12).
The second part of the chapter will be devoted to the study of the *polynomial partitioning theorem* over the plane. In particular, we will prove Theorem 2.2.5 and we will use it to give a proof of a cental result in incidence geometry, the Szemerédi-Trotter (Theorem 2.2.9).
Most of the results in this chapter have been extracted from the book of Guth [Gut16] and the paper by Kaplan, Matoušek and Sharir [KMS12]. Before proceeding to the main body of the chapter, let us introduce some notation.

Notation 2.0.1. Given any set of objects $\mathcal{S}$, we will denote by $|\mathcal{S}|$ the cardinality of $\mathcal{S}$.

Notation 2.0.2. We will denote by $\mathbb{F}$ any field. Let q be a prime power, we will let $\mathbb{F}_q$ stand for a finite field with q elements (unique up to isomorphism).

Notation 2.0.3. The $\mathbb{F}$-vector space of polynomials in $\mathbb{F}[x_1, \dots, x_n]$ of degree at most D will be denoted by $\mathbb{F}[x_1, \dots, x_n]_{\leq D}$.

Notation 2.0.4. Let $f = f(x, y) = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{R}[x, y]$ be a bivariate polynomial. The *zero set of f* is defined as follows:

$$Z(f) = \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}.$$

2.1 Siegel's lemma

We start by recalling the following standard result.

Lemma 2.1.1. *The dimension of* $\mathbb{F}[x_1,\ldots,x_n]_{\leq D}$ *is* $\binom{D+n}{n}$.

Proof. Consider a basis for $\mathbb{F}[x_1,\ldots,x_n]_{\leq D}$ given by monomials of the form $x_1^{d_1}\cdots x_n^{d_n}$ where $\sum_{i=1}^n d_i \leq D$. We can construct a bijection between the set of monomials of the previous form and the set of all possible strings of D $*$'s and n $|$'s. Indeed, given a monomial $x_1^{d_1}\cdots x_n^{d_n}$ we let the first d_1 elements of a string to be $*$'s. Then we add a $|$. The next elements of the string will be d_2 $*$'s followed by a single $|$. We continue until we have added the $*$'s corresponding to the exponent d_n and the nth $|$. To finish constructing the string we have to append $D-\sum_{i=1}^n d_i$ $*$'s.
Since the set of all possible strings of D $*$'s and n $|$'s has $\binom{D+n}{n}$ elements we conclude that the dimension of $\mathbb{F}[x_1,\ldots,x_n]_{\leq D}$ is $\binom{D+n}{n}$. □

Lemma 2.1.2 (Parameter counting argument)**.** *Let* $\mathcal{S}\subset\mathbb{F}^n$ *and* D *be a positive integer. If* $|\mathcal{S}|<\binom{D+n}{n}$*, then there is a nonzero polynomial* $P\in\mathbb{F}[x_1,\ldots,x_n]_{\leq D}$ *that vanishes on* $\mathcal{S}$.

Proof. Let $m:=|\mathcal{S}|$ and $s_1,\ldots,s_m$ be the elements of $\mathcal{S}$. Consider the map

$$\mu_{\mathcal{S}}:\mathbb{F}[x_1,\ldots,x_n]_{\leq D}\to\mathbb{F}^m \text{ given by } \mu_{\mathcal{S}}(P)=(P(s_1),\ldots,P(s_m)).$$

It is clear that $\mu_{\mathcal{S}}$ is a linear map and its kernel is the set of all polynomials in $\mathbb{F}[x_1,\ldots,x_n]_{\leq D}$ that vanish on $\mathcal{S}$. We want to see that $\ker(\mu)\neq\emptyset$. By Lemma 2.1.1 we know that $|\mathcal{S}|<\dim\mathbb{F}[x_1,\ldots,x_n]_{\leq D}$ therefore, by the rank-nullity theorem we obtain that $\dim\ker(\mu)=\dim\mathbb{F}[x_1,\ldots,x_n]_{\leq D}-m>0$. We conclude that there exists a nonzero polynomial $P\in\mathbb{F}[x_1,\ldots,x_n]_{\leq D}$ such that $P(\mathcal{S}_i)=0$ for all $1\leq i\leq m$. □

Corollary 2.1.3 (Siegel's lemma)**.** *Let* $n\geq 2$*. For any finite set* $\mathcal{S}\subset\mathbb{F}^n$*, there is a nonzero polynomial that vanishes on* $\mathcal{S}$ *with degree* $\leq n|\mathcal{S}|^{1/n}$.

Proof. Let D be the greatest integer smaller or equal than $n|\mathcal{S}|^{1/n}$. From this we deduce that

$$n|\mathcal{S}|^{1/n}<D+1\leq D+n \implies |\mathcal{S}|<\frac{(D+n)^n}{n^n}\leq\binom{D+n}{n},$$

where the last inequality is seen as follows. Observe that for $0<t<k\leq m$ we have the inequality $\frac{m}{k}\leq\frac{m-t}{k-t}$. This yields the bound

$$\left(\frac{m}{k}\right)^k\leq\frac{m}{k}\frac{m-1}{k-1}\cdots(m-k+1)=\binom{m}{k}.$$

Since $|\mathcal{S}|<\binom{D+n}{n}$ we can use Lemma 2.1.2 to find that there exists a nonzero polynomial $P\in\mathbb{F}[x_1,\ldots,x_n]$ of degree at most $D\leq n|\mathcal{S}|^{1/n}$ that vanishes on $\mathcal{S}$. □

Lemma 2.1.4 (Vanishing lemma)**.** *Let $P \in \mathbb{F}[x_1, \dots, x_n]$ with degree at most D. If P vanishes at $D+1$ points on a 1-dimensional affine subspace $l \subset \mathbb{F}^n$, then P vanishes at every point of l.*

Proof. Consider the parametrization of l by the map $\gamma : \mathbb{F} \to \mathbb{F}^n$ given by $\gamma(t) = at + b$, where $a, b \in \mathbb{F}^n$ and $a \neq 0$. Let $Q(t) := P(\gamma(t)) = P(at+b)$. Observe that $Q \in \mathbb{F}[x]$ and $\deg(Q) \leq D$. Since P vanishes at $D+1$ points of l, the polynomial Q has to vanish at $D+1$ values of t. Since Q is a polynomial in one variable of degree at most D vanishing at $D+1$ points it has to be the zero polynomial. Hence $P(at+b) = 0$ for all $t \in \mathbb{F}$ as we wanted to see. □

Theorem 2.1.5. *Let $\mathbb{K}$ be an infinite field and let $P \in \mathbb{K}[x_1, \dots, x_n]$. If $P(c_1, \dots, c_n) = 0$ for all $(c_1, \dots, c_n) \in \mathbb{K}^n$ then P is the zero polynomial.*

Proof. The case $n = 1$ is trivial since any polynomial $P \in \mathbb{K}[x]$ vanishes in a finite number of points. Suppose that $n > 1$ and that the result is true for all smaller values of n. Observe that we can write P as

$$P(x_1, \dots, x_n) = \sum_{j=0}^{N} P_j(x_1, \dots, x_{n-1}) x_n^j$$

where $P_j \in \mathbb{K}[x_1, \dots, x_{n-1}]$ and some $N > 0$. Choose $(c_1, \dots, c_{n-1}) \in \mathbb{K}^{n-1}$ and consider the polynomial $Q(x) := P(c_1, \dots, c_{n-1}, x_n) \in \mathbb{K}[x]$. Since P vanishes at every point of $\mathbb{K}^n$, Q has to be the zero polynomial. This means that $P_j(c_1, \dots, c_{n-1}) = 0$ for every j and every $(c_1, \dots, c_{n-1}) \in \mathbb{K}^{n-1}$. By induction hypothesis P_j is the zero polynomial for every j hence P is also the zero polynomial. □

The above result also works over finite fields. The proof is exactly the same, we only need to set $N = q - 1$.

Corollary 2.1.6. *Suppose that $P \in \mathbb{F}_q[x_1, \dots, x_n]$ and $\deg(P) \leq q-1$. If P vanishes at every point of $\mathbb{F}_q^n$, then P is the zero polynomial.*

With the results presented above we can prove the Nikodym and Kakeya problems over finite fields and the joints problem over $\mathbb{R}^3$. Let us start by the Nikodym problem.

Definition 2.1.7 (Finite field Nikodym set)**.** A set $\mathcal{N} \subset \mathbb{F}_q^n$ is called a *Nikodym set* if for each point $x \in \mathbb{F}_q^n$ there is a line $L(x)$ containing x such that $L(x) \setminus \{x\} \subset \mathcal{N}$.

Theorem 2.1.8. *Any Nikodym set $\mathcal{N} \subset \mathbb{F}_q^n$ contains at least $(10n)^{-n} q^n$ elements.*

Proof. Suppose that $|\mathcal{N}| < (10n)^{-n} q^n$. By Corollary 2.1.3 there is a nonzero polynomial $P \in \mathbb{F}_q[x_1, \dots, x_n]$ that vanishes on $\mathcal{N}$ with $\deg(P) \leq n|N|^{1/n} < \frac{1}{10} q < q - 1$.

Claim 2.1.9. *The polynomial P vanishes at every point of $\mathbb{F}_q^n$.*

Proof. Let $x \in \mathbb{F}_q^n$. Since $\mathcal{N}$ is a Nikodyn set there exists a line $L(x)$ such that $x \in L(x)$ and $L(x) \setminus \{x\} \subset \mathcal{N}$. Since P vanishes on $\mathcal{N}$, P vanishes at $\geq q-1$ points of $L(x)$. Moreover $\deg(P) < q-1$, therefore we can apply Lemma 2.1.4 and conclude that P vanishes at all $L(x)$ and in particular at x. □

Applying Corollary 2.1.6 we obtain that P is the zero polynomial which is a contradiction. □

We continue by proving the well-known finite field Kakeya problem.

Definition 2.1.10 (Finite field Kakeya set)**.** A set $K \subset \mathbb{F}_q^n$ is a *Kakeya set* if for every $a \in \mathbb{F}_q^n \setminus \{0\}$ there exists a point $b \in \mathbb{F}_q^n$ such that the line $\{b + at \mid t \in \mathbb{F}_q\}$ is contained in $\mathcal{K}$.

Theorem 2.1.11. *Let $\mathcal{K} \subset \mathbb{F}_q^n$ be a Kakeya set. Then*

$$|\mathcal{K}| \geq (10n)^{-n} q^n.$$

Proof. Suppose that $|\mathcal{K}| < (10n)^{-n} q^n$. By Corollary 2.1.3 there is a nonzero polynomial $P \in \mathbb{F}_q[x_1, \ldots, x_n]$ that vanishes on $\mathcal{K}$ with $D := \deg P \leq n|\mathcal{K}|^{1/n} < \frac{1}{10}q < q$. Let P_D be the terms of degree D of P and let Q be the terms of lower degree. We can write $P = P_D + Q$. Observe that since P has degree D the polynomial P_D is nonzero and homogeneous of degree D.
Let $a \in \mathbb{F}_q^n$ be a nonzero vector. Since $\mathcal{K}$ is a Kakeya set we can find $b \in \mathbb{F}_q^n$ such that the line $l := \{b + at \mid t \in \mathbb{F}_q\}$ is contained in $\mathcal{K}$. Consider the polynomial $R(t) := P(b + at)$. This polynomial has degree $\leq D < q$ and vanishes at every value of t since P vanishes at every point of $\mathcal{K}$. Therefore R is the zero polynomial. This means that every coefficient of R is zero. Observe that the coefficient of t^D in R is exactly $P_D(a)$. Therefore $P_D(a)$ vanishes for all $a \in \mathbb{F}_q^n \setminus \{0\}$. Since P_D is homogeneous of degree $D \geq 1$, P_D also vanishes at 0 which means that P_D vanishes at every point of $\mathbb{F}_q^n$. Since $D < q$ we can apply Corollary 2.1.6 which tells us that P_D is the zero polynomial giving a contradiction. □

We now turn our attention to a problem in the Euclidean space $\mathbb{R}^3$. Let $\mathcal{L}$ be a set of lines in $\mathbb{R}^3$. A joint of $\mathcal{L}$ is a point which lies in at least three non-coplanar lines of $\mathcal{L}$. The joints problem asks what is the maximum number of joints that are determined by a set of L lines.

Theorem 2.1.12. *Let $\mathcal{L}$ be a set of L lines in $\mathbb{R}^3$ and let $\mathcal{J}$ be the set of J joints determined by $\mathcal{L}$. Then*

$$J \leq 10L^{3/2}.$$

Proof. We start by proving the following claim:

Claim 2.1.13. *Let a be a joint of $\mathcal{L}$. If a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ vanishes on the lines of $\mathcal{L}$ then the gradient $\nabla F = (\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z})$ vanishes at a.*

Proof. Since $a \in \mathcal{J}$ we can find three lines l_1, l_2, l_3 intersecting at a. Let b be a unit vector in the direction of l_1. Then we can parametrize l_1 by $a + bt$ with $t \in \mathbb{R}$. For a sufficiently small value of t, we have the expansion

$$F(a + bt) = F(a) + t\nabla F(a) \cdot b + O(t^2).$$

Since F vanishes at every line of $\mathcal{L}$ and $a \in l_1$ we obtain that $\nabla F(a) \cdot b = 0$. That is, $\nabla F(a)$ is orthogonal to the directional vector of l_1. By the same reasoning we find that $\nabla F(a)$ is also orthogonal to the directional vectors of l_2 and l_3. Since the lines l_1, l_2, l_3 are non-coplanar their directional vectors form a basis of the whole space $\mathbb{R}^3$ hence the only possibility is that $\nabla F(a) = 0$. $\square$

We will also need to know that there exists some line that contain few of the joints.

Claim 2.1.14. *If $\mathcal{L}$ is a set of lines in $\mathbb{R}^3$ that determines $J \neq 0$ joints, then one of the lines contains at most $3J^{1/3}$ joints.*

Proof. We argue by contradiction. Suppose that every line of $\mathcal{L}$ has more than $3J^{1/3}$ joints. By Corollary 2.1.3 there exists a polynomial P of degree at most $3J^{1/3}$ vanishing at the joints determined by $\mathcal{L}$. Hence, by Lemma 2.1.4 the polynomial P vanishes on every line of $\mathcal{L}$. Since P is a smooth function we can apply Claim 2.1.13 to find that $\nabla P(p) = 0$ for every joint p of $\mathcal{L}$. This means that the derivatives of P vanish at each joint of $\mathcal{L}$. Since the derivatives of P have lower degree than P and P is a polynomial of the lowest possible degree that vanishes at all the joints of $\mathcal{L}$ then all the derivatives must be zero. Hence, the polynomial P must be constant. If this is the case then the set of lines $\mathcal{L}$ does not determine any joint which is a contradiction. $\square$

Let $J(L)$ denote the maximum number of joints that can be formed by L lines and let $\mathcal{L}$ be a set of L lines. By the claim we know that one of the lines contains at most $3J(L)^{1/3}$ of the joints. The number of joints not on this line is at most $J(L-1)$. We obtain the following bound:

$$J(L) \leq J(L-1) + 3J(L)^{1/3}.$$

If we iterate:

$$J(L) \leq J(L-1) + 3J(L)^{1/3} \leq J(L-2) + 2 \cdot 3J(L)^{1/3} \leq \cdots \leq L \cdot 3J(L)^{1/3}.$$

This implies $J(L)^{2/3} \leq 3L$, thus $J(L) \leq \sqrt{27}L^{3/2} < 10L^{3/2}$, as we wanted to see. $\square$

2.2 Polynomial partitioning over $\mathbb{R}^2$

Let $\mathcal{S} \subset \mathbb{R}^2$ be a set of S points and let $\mathcal{L} \subset \mathbb{R}$ be a set of L, the *set of incidences between $\mathcal{S}$ and $\mathcal{L}$* is defined as follows:

$$I(\mathcal{S}, \mathcal{L}) = \{(p, l) \in \mathcal{S} \times \mathcal{L} \mid p \in l\}.$$

In this section we will be interested in obtaining a tight upper bound for

$$I(\mathcal{S}, \mathcal{L}) := |\mathcal{I}(\mathcal{S}, \mathcal{L})|.$$

As we have already seen in the introduction, this is given by the Szemerédi-Trotter theorem, Theorem 2.2.9. There are many proofs of this result, see for instance [Szé97] for a proof based on graph theory and [ST83] for the original proof. In this section we give a proof of Theorem 2.2.9 using the polynomial method.
As opposed to the proofs presented in the last section, we are now interested in finding a polynomial which partitions a set of points in the plane instead of finding a polynomial vanishing on it. See Figure 2.1 for a very simple illustration of this.

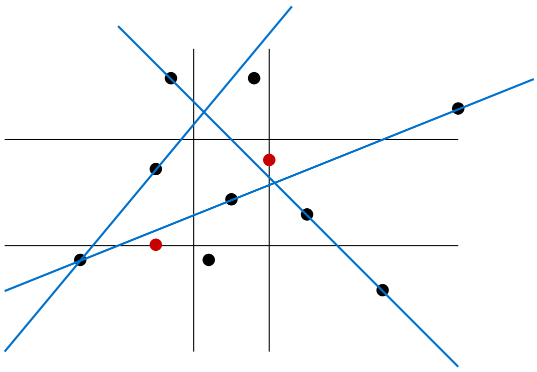

Figure 2.1: Given a finite set of lines (blue lines) and a finite set of points we want to partition $\mathbb{R}^2$ by the zero set of a polynomial $Z(P)$ (black lines) in such a way that few points lie within each connected component of $\mathbb{R}^2 \setminus Z(P)$. Clearly, it may happen that some points fall within $Z(P)$ (red points).

To find such polynomial we will make use of a result from topology, *the ham sandwich theorem* proven by Stone and Tukey in the 1940s.

Theorem 2.2.1 (Ham sandwich theorem, Stone-Tukey [ST42]). *Let $\mathcal{U}_1, \dots, \mathcal{U}_n \subset \mathbb{R}^n$ be finite volume open sets. Then there is a hyperplane which bisects each set $\mathcal{U}_i$.*

We will need the following discrete version of the ham sandwich theorem.

Definition 2.2.2. We say that a polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ *bisects* a finite set $\mathcal{S} \subset \mathbb{R}^n$ if $f > 0$ in at most $\lfloor |\mathcal{S}|/2 \rfloor$ points of $\mathcal{S}$ and $f < 0$ in at most $\lfloor |\mathcal{S}|/2 \rfloor$ points of $\mathcal{S}$.

Theorem 2.2.3 (Discrete ham sandwich theorem). *Let $\mathcal{S}_1, \dots, \mathcal{S}_n \subset \mathbb{R}^n$ be finite sets. Then there is a hyperplane which bisects each set $\mathcal{U}_i$.*

In order to obtain a partitioning theorem we will use the following variation of the above theorem, which allow us to bisect a finite set of points not only with a hyperplane but with the zero set of a polynomial of bounded degree.

Theorem 2.2.4 (Polynomial ham-sandwich theorem)**.** *Let $\mathcal{S}_1, \ldots, \mathcal{S}_s \subset \mathbb{R}^2$ be finite sets, and let D be an integer such that $\binom{D+2}{2} - 1 \geq s$. Then there exists a nonzero polynomial $P \in \mathbb{R}[x, y]$ of degree at most D that simultaneously bisects all the sets $\mathcal{S}_i$.*

The polynomial ham sandwich theorem allows us to find an algebraic surface bisecting a finite set of points. This gives us two finite sets of points which again can be partitioned by a polynomial provided by the ham sandwich theorem. If we iterate this process and we keep track of the degrees of the polynomials that we find we get the following theorem.

Theorem 2.2.5 (Polynomial Partitioning)**.** *For every finite set $\mathcal{S} \subset \mathbb{R}^2$ of points and every $r > 1$ there exists a nonzero polynomial $f \in \mathbb{R}[x, y]$ of degree at most $O(\sqrt{r})$ such that every connected component of $\mathbb{R}^2 \setminus Z(f)$ contains*

$$\lesssim \frac{|\mathcal{S}|}{r}$$

points of $\mathcal{S}$.

Proof. By Theorem 2.2.4 we can find a nonzero polynomial $f_1 \in \mathbb{R}[x, y]$ with $\deg(f_1) \leq \sqrt{2}\sqrt{2} = 2$ bisecting $\mathcal{S}$. Consider the sets

$$Q^+ := \{x \in S \mid f_1(x) > 0\} \text{ and } Q^- := \{x \in S \mid f_1(x) < 0\}.$$

Observe that $|Q^+| \leq \lfloor |\mathcal{S}|/2 \rfloor$ and $|Q^-| \leq \lfloor |\mathcal{S}|/2 \rfloor$. Let us define the set $S_1 = \{Q^+, Q^-\}$. We can apply Theorem 2.2.4 again on the elements of S_1. Doing so we find a polynomial $f_2 \in \mathbb{R}[x, y]$ with $\deg f_2 \leq \sqrt{2}\sqrt{3} < 2\sqrt{2}$ bisecting Q^+ and Q^-. Let S_2 be the set whose elements are the sets obtained from this bisection. Proceeding inductively we can suppose that we have constructed sets $\mathcal{S}_i$ with at most 2^i sets. Applying Theorem 2.2.4 we find a polynomial $f_i \in \mathbb{R}[x, y]$ with $\deg(f_i) \leq \sqrt{2}\sqrt{2^i}$ bisecting all the elements of $\mathcal{S}_i$. As we did before, for each $Q \in \mathcal{S}_i$ we define

$$Q^+ := \{x \in S \mid f_i(x) > 0\} \text{ and } Q^- := \{x \in S \mid f_i(x) < 0\}$$

and let $S_{i+1} := \bigcup_{Q \in \mathcal{S}_i} \{Q^+, Q^-\}$. This process finishes at $t = \lceil \log_2 r \rceil$ where we find a polynomial $f_t \in \mathbb{R}[x, y]$ with $\deg(f_t) \leq \sqrt{2}\sqrt{r}$ and we obtain the set S_t with at most r elements. Let $f = f_1 f_2 \cdots f_t$. By construction of f no component of $\mathbb{R}^2 \setminus Z(f)$ can contain points of two different sets of S_t therefore each component contains $\lesssim \frac{|\mathcal{S}|}{r}$ elements of $\mathcal{S}$. We conclude by bounding the degree of f.

$$\begin{aligned}
\deg(f) &= \sum_{i=1}^{t} \deg(f_i) \leq \sqrt{2} \sum_{i=1}^{t} \sqrt{2^i} = \sqrt{2} \sum_{i=1}^{t} \sqrt{2}^i = \sqrt{2} \frac{\sqrt{2}(1 - \sqrt{2})}{1 - \sqrt{2}} \\
&= \frac{2}{1 - \sqrt{2}} + \frac{2 \cdot 2^{t/2}}{\sqrt{2} - 1} < \frac{2 \cdot 2^{t/2}}{\sqrt{2} - 1} \leq c\sqrt{r}
\end{aligned}$$

where $c = \frac{\sqrt{2}}{\sqrt{2}-1}$. □

We finish this chapter by proving the Szemerédi-Trotter theorem, Theorem 2.2.9. This proof will relay on the two following consequences of Bezout's theorem.

Lemma 2.2.6. *If l is a line in $\mathbb{R}^2$ and $f \in \mathbb{R}[x,y]$ is of degree at most D, then either $l \subseteq Z(f)$ or $|l \cap Z(f)| \leq D$.*

Proof. Consider the parametrization of l by the map $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (a_1 t + b_1, a_2 t + b_2)$, where $a_1, b_1, a_2, b_2 \in \mathbb{R}$. Observe that the points of $l \cap Z(f)$ are the roots of $g := f \circ \gamma$. If g is identically zero then $l \subseteq Z(f)$. Otherwise, since $\deg(g) = \deg(f) \leq D$ we obtain that $|l \cap Z(f)| \leq D$. □

Lemma 2.2.7. *If $f \in \mathbb{R}[x,y]$ is nonzero and of degree at most D, then $Z(f)$ contains at most D distinct lines.*

Proof. Suppose that $Z(f)$ contains k lines $l_1, \dots, l_k$. We want to see that $k \leq D$. Take a point $p \in \mathbb{R}^2 \setminus Z(f)$ and consider the line l passing through p, not parallel to any l_i for $1 \leq i \leq k$ and without crossing any intersection point $l_i \cap l_j$ for $1 \leq i < j \leq k$. Therefore l has k intersections with $\bigcup_{i=1}^{k} l_i$ and it is not contained in $Z(f)$. By Lemma 2.2.6 we know that $k = |l \cap Z(f)| \leq D$. □

When using the polynomial partitioning theorem to tackle a problem in incidence geometry we need a weak bound to apply to the incidences within the cells that we will create. In this case, these bound is given by the next lemma.

Lemma 2.2.8. *Let $\mathcal{S}$ be a set of S points and let $\mathcal{L}$ be a set of L lines in the plane, then*

$$I(\mathcal{S}, \mathcal{L}) \leq L + S^2 \quad \text{and} \quad I(\mathcal{S}, \mathcal{L}) \leq S + L^2.$$

Proof. We start by proving the inequality $I(\mathcal{S}, \mathcal{L}) \leq L + S^2$. Observe that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ where $\mathcal{L}_1$ is the set containing the lines of $\mathcal{L}$ which are incident to at most one point of $\mathcal{S}$ and $\mathcal{L}_2$ is the set of lines of $\mathcal{L}$ which contain at least two points of $\mathcal{S}$. We clearly obtain the following bound

$$I(\mathcal{S}, \mathcal{L}_1) \leq |\mathcal{L}_1| \leq L.$$

Notice that there are at most $S - 1$ lines passing through a point $p \in \mathcal{S}$ and some other point of $\mathcal{S}$. This means that any point $p \in \mathcal{S}$ can have at most $S-1$ incidences with the lines of $\mathcal{L}_2$. Therefore, we get the bound

$$I(\mathcal{S}, \mathcal{L}_2) \leq S(S-1) \leq S^2,$$

Since

$$I(\mathcal{S}, \mathcal{L}) = I(\mathcal{S}, \mathcal{L}_1) + I(\mathcal{S}, \mathcal{L}_1)$$

we get the result. The other inequality follows from planar duality. □

Theorem 2.2.9 (Szemerédi-Trotter)**.** *Let $\mathcal{S} \subset \mathbb{R}^2$ be a set of S distinct points and let $\mathcal{L}$ be a set of L distinct lines in the plane, then*

$$I(\mathcal{S}, \mathcal{L}) \lesssim S^{2/3} L^{2/3} + S + L.$$

Proof. Suppose that $S \leq L$. Notice that if $\sqrt{L} > S$ then the result follows by Lemma 2.2.8 therefore we may also assume that $\sqrt{L} \leq S$. Let

$$r = \frac{S^{4/3}}{L^{2/3}}.$$

By Theorem 2.2.5 we know that there exists some polynomial $f \in \mathbb{R}[x, y]$ such that every connected component of $\mathbb{R}^2 \setminus Z(f)$ contains $\lesssim \frac{L^{2/3}}{S^{1/3}}$ points of $\mathcal{S}$ and

$$D := \deg(f) \lesssim \sqrt{r} = \frac{S^{2/3}}{L^{1/3}}.$$

Let $C_1, \ldots, C_s$ be the connected components of $\mathbb{R}^2 \setminus Z(f)$. Define the sets

$$\mathcal{S}_0 := Z(f) \cap \mathcal{S} \text{ and } \mathcal{S}_i := C_i \cap \mathcal{S} \text{ for } i = 1, \ldots, s.$$

Let $\mathcal{L}_0 = \{l \in \mathcal{L} \mid l \subset Z(f)\}$. By Lemma 2.2.7 we obtain the bound $|\mathcal{L}_0| \lesssim \frac{S^{2/3}}{L^{1/3}}$. With all of the above we can count the incidences in the following way:

$$I(\mathcal{S}, \mathcal{L}) = \sum_{i=1}^{s} I(\mathcal{S}_i, \mathcal{L}) + I(\mathcal{S}_0, \mathcal{L} \setminus \mathcal{L}_0) + I(\mathcal{S}_0, \mathcal{L}_0). \tag{2.2.1}$$

That is, we will count the incidences within the cells and the incidences within the zero set of f. Let us start with the former. Let

$$\mathcal{L}_i = \{l \in \mathcal{L} \mid \exists\, p \in \mathcal{S}_i, p \in l\}$$

for $i = 1, \ldots, s$. It is clear that we can ignore the lines that do not intersect points within the cells, therefore $\sum_{i=1}^{s} I(\mathcal{S}_i, \mathcal{L}) = \sum_{i=1}^{s} I(\mathcal{S}_i, \mathcal{L}_i)$. Using the bound provided by Lemma 2.2.8 we get

$$\sum_{i=1}^{s} I(\mathcal{S}_i, \mathcal{L}_i) \leq \sum_{i=1}^{s} (|\mathcal{L}_i| + |\mathcal{S}_i|^2) = \sum_{i=1}^{s} |\mathcal{L}_i| + \sum_{i=1}^{s} |\mathcal{S}_i|^2. \tag{2.2.2}$$

By Lemma 2.2.6 we know that a line l of $\mathcal{L}$ can intersect $Z(f)$ in at most D points. Hence, l can only intersect $D + 1$ connected components of $\mathbb{R} \setminus Z(f)$. This gives us the following bound.

$$\sum_{i=1}^{s} |\mathcal{L}_i| \leq (D + 1)L \lesssim S^{2/3} L^{2/3} + L. \tag{2.2.3}$$

Moreover using the bound obtained from the polynomial partitioning theorem we know that $|\mathcal{S}_i| \lesssim \frac{L^{2/3}}{S^{1/3}}$ therefore

$$\sum_{i=1}^{s} |\mathcal{S}_i|^2 \leq (\max_i |\mathcal{S}_i|) \sum_{i=1}^{s} |\mathcal{S}_i| \lesssim \frac{L^{2/3}}{S^{1/3}} S = L^{2/3} S^{2/3}. \tag{2.2.4}$$

Therefore, by inequality (2.2.3) we obtain the following bound.

$$\sum_{i=1}^{s} I(\mathcal{S}_i, \mathcal{L}) \lesssim S^{2/3} L^{2/3} + L. \tag{2.2.5}$$

Let us finish by counting the incidences within $Z(f)$. Using Lemma 2.2.7 we get

$$I(\mathcal{S}_0, \mathcal{L}_0) \leq |\mathcal{S}_0||\mathcal{L}_0| \leq SD \lesssim S\frac{S^{2/3}}{L^{1/3}} \leq L\frac{S^{2/3}}{L^{1/3}} = S^{2/3} L^{2/3}. \tag{2.2.6}$$

Moreover we have the inequality

$$I(\mathcal{S}_0, \mathcal{L} \setminus \mathcal{L}_0) \leq |\mathcal{L} \setminus \mathcal{L}_0| D \leq LD \lesssim L\frac{S^{2/3}}{L^{1/3}} = S^{2/3} L^{2/3}. \tag{2.2.7}$$

From inequalities (2.2.1), (2.2.5), (2.2.6) and (2.2.7) we conclude that

$$I(\mathcal{S}, \mathcal{L}) \lesssim S^{2/3} L^{2/3} + L$$

This concludes the proof for the case $S \leq L$. For the other case we can use standard planar duality to interchange the roles of $\mathcal{S}$ and $\mathcal{L}$ to obtain the bound

$$I(\mathcal{S}, \mathcal{L}) \lesssim S^{2/3} L^{2/3} + S,$$

which completes the proof. □

Chapter 3

Preliminary concepts and results

In this chapter we begin the work to extend the polynomial method to algebraic varieties of arbitrary dimension. We will start by introducing some notation and recalling standard facts from algebraic geometry will be used in most of the remaining chapters.
We will continue by introducing the concept of *partial degree* of an irreducible algebraic variety which will play a crucial role in this work. We will end the chapter by stating a collection of results about *Hilbert functions*.
Without taking into account the first section, the vast majority of the results of these chapter are extracted from the paper of Walsh [Wal18, Sections 3-5].
We will use the following notation.

Notation 3.0.1. Given parameters $a_1, \ldots, a_r$ we will write $X \lesssim_{a_1,\ldots,a_r} Y$ or $X = O_{a_1,\ldots,a_r}(Y)$ when there exists a constant C depending only on $a_1, \ldots, a_r$ such that $X \leq CY$. We will also write $X \sim_{a_1,\ldots,a_r} Y$ when $X \lesssim_{a_1,\ldots,a_r} Y \lesssim_{a_1,\ldots,a_r} X$.

Notation 3.0.2. Given polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$ we define their zero set as

$$Z(f_1, \ldots, f_r) := \{x \in \mathbb{C}^n \ : \ f_1(x) = \cdots = f_r(x) = 0\}.$$

Notation 3.0.3. We write $I(V)$ for the ideal of an algebraic variety V and write $I_{\mathbb{R}}(V) \subseteq I(V)$ for its subset of real polynomials. We will also write $V(\mathbb{R})$ for the real points of V.

3.1 Standard results from algebraic geometry

Recall that the following class of topological spaces includes all varieties.

Definition 3.1.1 (Noetherian topological space)**.** A topological space is called *noetherian* if for any sequence $Y_1 \supseteq Y_2 \supseteq \cdots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \cdots$. This condition is called the *descending chain condition.*

Definition 3.1.2 (Dimension of an affine variety)**.** Let X be a topological space. The *dimension* of X, $\dim(X)$ is the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X. We define the *dimension* of an affine variety to be its dimension as a topological space.

We will use the following standard fact about hypersurfaces.

Proposition 3.1.3. *Let $\mathbb{F}$ be a field. A variety Y in the affine space $\mathbb{A}^n$ over $\mathbb{F}$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single nonconstant irreducible polynomial in $\mathbb{F}[x_1, \dots, x_n]$.*

Definition 3.1.4 (Degree of a variety)**.** The *degree* of an d-dimensional algebraic variety $V \subseteq \mathbb{C}^n$ is defined as the number of points of intersection of V with a sufficiently general linear space of dimension $n-d$. If V is an algebraic set and $V_1, \dots, V_r$ its irreducible components we write $\deg(V) = \sum_{i=1}^{r} \deg(V_i)$.

In order to establish useful estimates for the degree of the varieties that we will encounter we will need to extensively use the following formulation of the well known *Bezout's inequality.*

Lemma 3.1.5 (Bezout's inequality)**.** *Let $W \subseteq \mathbb{C}^n$ be an algebraic variety and $f_1, \dots, f_s \in \mathbb{C}[x_1, \dots, x_n]$ polynomials. Write $Z_1, \dots, Z_r$ for the irreducible components of $Z(f_1, \dots, f_s) \cap W$. Then*

$$\sum_{i=1}^{r} \deg(Z_i) \leq \deg(W) \prod_{j=1}^{s} \deg(f_j).$$

Proof. Let $C_1, \dots, C_t$ be the irreducible components of W. Notice that

$$\deg(W \cap Z(f)) \leq \sum_{i=1}^{t} \deg(C_i \cap Z(f)).$$

for any polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$.
If $C_i \subseteq Z(f)$ then we get $\deg(C_i \cap Z(f)) = \deg(C_i) \leq \deg(C_i)\deg(f)$. On the other hand, if $C_i \nsubseteq Z(f)$ we can apply lemma 7.7 from [Har77] and we obtain $\deg(C_i \cap Z(f)) \leq \deg(C_i)\deg(f)$. We conclude that

$$\deg(W \cap Z(f)) \leq \sum_{i=1}^{r} \deg(C_i)\deg(f) = \deg(W)\deg(f).$$

Using the above inequality we obtain the bound

$$\begin{aligned} \deg(W \cap Z(f_1, \dots, f_s)) &\leq \deg((W \cap Z(f_1, \dots, f_{s-1})) \cap Z(f_s)) \\ &\leq \deg(W \cap Z(f_1, \dots, f_{s-1}))\deg(f_s). \end{aligned}$$

Iterating from f_{s-1} to f_1 we get the result. □

Definition 3.1.6. (Locally closed) Let X be a topological space and Z a subset of X. We say Z is *locally closed in* X if, for any point $z \in Z$, there exists an open neighbourhood U of z in X such that $U \cap Z$ is closed in U. It is easy to see that Z is locally closed in X if and only if it is expressible as the intersection of an open set in X and a closed set in X.

Definition 3.1.7. Let X be a noetherian space. We say a subset Z of X is a *constructible set in* X if Z is a finite union of locally closed sets in X:

$$Z = \bigcup_{i=1}^{m} U_i \cap F_i$$

where U_i is open and F_i is closed.

Theorem 3.1.8 (Chevalley)**.** *Let $\phi : X \to Y$ be a morphism of varieties. Then ϕ maps constructible sets to constructible sets.*

Lemma 3.1.9. *Let $t \subseteq \mathbb{C}^n$ be an l-dimensional irreducible variety and let H be a finite family of irreducible varieties of $\mathbb{C}^n$ of dimension $l-1$. Then there exists some irreducible variety $h \subseteq t$ of dimension $l-1$ with $\deg(h) \leq \deg(t)$ and $h \notin H$.*

Proof. Consider a generic linear map $\pi : \mathbb{C}^n \to \mathbb{C}^l$. Since t is an irreducible algebraic variety of dimension l we know that t is in Noether position with respect to π, that is, $\pi_{|t} : t \to \mathbb{C}^l$ is finite. Thus $\pi(t) = \mathbb{C}^l$.
Moreover, $\pi(H)$ is a constructible set by Theorem 3.1.8. This means that $\overline{\pi(H)}$ is an algebraic set of dimension $\leq l-1$. Now, take $L \subset C^l$ such that $\dim(L) = l-1$ and $Z \not\subset L$ for every irreducible component Z of $\overline{\pi(H)}$.
We know that $\pi^{-1}(L)$ is an affine subspace of dimension $n-1$. Observe that $t \not\subset \pi^{-1}(L)$ by our choice of L. Therefore we can take $h = \pi^{-1}(L) \cap t$. □

3.2 Partial degree of an algebraic variety

In this section we introduce the definition of *partial degree* of an algebraic variety $V \subseteq \mathbb{C}^n$. From this definition we will prove estimates relating the degree of V and its partial degree.
More precisely, we want to find bounds for the degree of an irreducible algebraic variety in terms of its partial degree. This will be accomplished by Corollary 3.2.8, Theorem 3.2.11 and Corollary 3.2.13.
A crucial role in the proof of Theorem 3.2.11 will be played by Theorem 4.0.3, a result that will be discussed in detail in chapter 4.

Definition 3.2.1 (Partial degree)**.** For an irreducible algebraic variety $V \subseteq \mathbb{C}^n$ and every $1 \leq i \leq n - \dim(V)$ we let $\delta_i(V)$ stand for the minimal integer δ for which we can find a finite set of polynomials $g_1, \ldots, g_t \in \mathbb{C}[x_1, \ldots, x_n]$ of degree at most δ such that $V \subseteq Z(g_1, \ldots, g_t)$ and the highest dimension of an irreducible component of $Z(g_1, \ldots, g_t)$ containing V is equal to $n - i$. Sometimes we will set $\delta_{n-\dim(V)} = \delta(V)$ and call this the *partial degree* of V. By convention we write $\delta_0(V) = 0$ and $\delta_i(V) = \infty$ for every $i > n - \dim(V)$.

Lemma 3.2.2. *For every variety V we have $\delta_i(V) \geq \delta_{i-1}(V)$ for every i.*

Proof. By definition of $\delta_i(V)$ there exist polynomials $g_1, \ldots, g_t \in \mathbb{C}[x_1, \ldots, x_n]$ such that the highest dimension of an irreducible component of $Z(g_1, \ldots, g_s)$ containing V is $n-i$. If we take the subset of polynomials $g_i, \ldots, g_r$ that do not vanish on the irreducible components of dimension $n-i$ containing V then the highest dimension of an irreducible component of $Z(g_i, \ldots, g_r)$ containing V is $n-i+1$ which means that $\delta_i(V) \geq \delta_{i-1}(V)$. □

We will make frequent use of the following quantities associated to the partial degrees of a variety.

Notation 3.2.3. Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d, we write

$$\Delta_i(V) := \max\left\{\frac{\deg(V)}{\delta_{i+1}(V)\cdots\delta_{n-d}(V)}, 1\right\} \quad \text{and} \quad \Pi_i(V) := \delta_i(V)^{n-i}\Delta_i(V).$$

Definition 3.2.4 (Minimal variety)**.** Let $V \subseteq \mathbb{C}^n$ be an irreducible variety. We say an irreducible variety $V' \subset \mathbb{C}^n$ containing V is an $(n-s)$-*minimal variety of* V if $\dim(V') = n-s$ and every polynomial of $I(V) \setminus I(V')$ has degree at least $\delta_{s+1}(V)$.

Notice that the definition of a $(n-s)$-minimal variety V' of an irreducible variety $V \subseteq \mathbb{C}^n$ implies that the ideal $I(V')$ has to contain all polynomials of $I(V)$ of degree $< \delta_{s+1}(V)$. Moreover, V' is not necessarily unique since it can also contain polynomials of degree $\geq \delta_{s+1}(V)$.

Lemma 3.2.5. *Let $V_1, \ldots, V_r$ be subsets of $\mathbb{C}^n$ and let $S \subseteq V_1 \cap \cdots \cap V_r$. Let $f_1, \ldots, f_r$ be polynomials such that $f_i \in I(\mathcal{S})$ and $f_i \notin I(V_i)$ for all $1 \leq i \leq r$. Then, there is a nontrivial linear combination $f = c_1 f_1 + \cdots + c_r f_r$, with real coefficients, such that $f \in I(\mathcal{S})$ and $f \notin I(V_i)$ for all $1 \leq i \leq r$.*

Proof. Let $\mathbb{F}$ be any field. Take a vector $c = (c_1, \ldots, c_r) \in \mathbb{F}^r$ and define the following polynomial

$$f_c(x) := \sum_{i=1}^{r} c_i f_i.$$

It is clear that $f_c \in I(\mathcal{S})$. Let us see that $f_c \notin I(V_i)$ for every i. For every i, take an element $p_i \in V_i$ such that $f_i(p_i) \neq 0$ and consider the polynomial

$$P(c) := \prod_{i=1}^{r} f_c(p_i) \in \mathbb{C}[c].$$

Clearly P is not the zero polynomial. Therefore by Theorem 2.1.5 we know that we can find $c \in \mathbb{R}^r$ such that $P(c) \neq 0$. Therefore $f_c(p_i) \neq 0$ for every i. Thus it suffices to take $f := f_c$. □

Lemma 3.2.6. *Let $V \subseteq \mathbb{C}^n$ be an irreducible variety of dimension d. Then, there exist polynomials $P_1, \ldots, P_{n-d}$ defined recursively such that, for every $1 \leq i \leq n-d$, $\deg(P_i) = \delta_i(V)$, P_i is a polynomial of the smallest possible degree such that the maximal dimension of an irreducible component of $Z(P_1, \ldots, P_i)$ containing V is $n-i$ and $Z(P_1, \ldots, P_i)$ contains an $(n-i)$-minimal variety of V.*

Proof. This result follows from the definition of partial degree. In the case where $i = 1$ we know that we can find polynomials $g_1, \ldots, g_s \in \mathbb{C}[x_1, \ldots, x_n]$ such that $\deg(g_i) \leq \delta_1(V)$, $V \subseteq Z(g_1, \ldots, g_s)$ and the highest dimension of an irreducible component of $Z(g_1, \ldots, g_s)$ containing V is $n-1$. It suffices to take one of the polynomials g_i such that $V \subseteq Z(g_i)$ since $\dim Z(g_i) = n-1$.
Let us iterate. Suppose that we have constructed $i-1$ polynomials $P_1, \ldots, P_{i-1}$ satisfying the hypotheses of the lemma. Let $V_1, \ldots, V_r$ be the irreducible components of $Z(P_1, \ldots, P_{i-1})$ such that $V \subseteq V_j$ and $\dim(V_j) = n-(i-1)$ for all $1 \leq j \leq r$. By definition of partial degree, there exist polynomials $g_1, \ldots, g_s \in \mathbb{C}[x_1, \ldots, x_n]$ of degree at most $\delta_i(V)$ such that the maximal dimension of an irreducible component of $Z(g_1, \ldots, g_s)$ containing V is $n-i$. Since $\dim(V_j) > n-i$ and $V \subseteq V_j$ it must be that $V_j \not\subseteq Z(g_1, \ldots, g_s)$. Therefore, for each V_j we can find a polynomial f_j of degree at most $\delta_i(V)$ such that $f_j \in I(V)$ and $f_j \notin I(V_j)$. Then by Lemma 3.2.5 we find a linear combination f with real coefficients of the polynomials f_j such that $f \in I(V)$ and $f \notin I(V_j)$ for every $1 \leq j \leq r$. Let us see that we can take $P_i = f$.

Claim 3.2.7. $\deg(f) = \delta_i(V)$.

Proof. Suppose that $\deg(f) < \delta_i(V)$. Let $s \leq i$ be the smallest integer with $\delta_s(V) = \delta_i(V)$ and let $W_1, \ldots, W_t$ be the irreducible components of $Z(P_1, \ldots, P_{s-1})$ containing V such that $\dim(W_m) = n-s+1$ for all $1 \leq m \leq t$. Notice that each W_m contains one of the previous V_j since $n-(i-1) \leq n-(s-1)$. This means that $f \notin I(W_m)$ for every m. Hence we find a contradiction with the definition of partial degree since the maximal dimension of an irreducible component of $Z(P_1, \ldots, P_{s-1}, P_i)$ containing V is $n-s$ while $\deg(P_1), \ldots, \deg(P_{s-1}), \deg(P_i) < \delta_s(V)$. □

Notice that we have shown that there is some V_j with every polynomial of $I(V) \setminus I(V_j)$ having degree at least $\delta_i(V)$, that is, V_j is a $n-(i-1)$ minimal variety of V. Indeed, suppose that for every V_j there exists a polynomial $f_j \in I(V) \setminus I(V_j)$ such that $\deg(f_j) < \delta_i(V)$. Applying Lemma 3.2.5 we find a linear combination such that $f \in I(V) \setminus I(V_j)$ for every j with $\deg(f) < \delta_i(V)$. However in the proof of the claim we have seen that this enters in contradiction with the definition of partial degree.

This proves that we can take $P_i = f$. At the end of the iteration, when $i = n-d$ we need to find a d-minimal variety in $Z(P_1, \ldots, P_{n-d})$ but we can take V as this variety, proving the result. □

Corollary 3.2.8. *Every irreducible algebraic variety V of dimension d satisfies*

$$\deg(V) \leq \prod_{i=1}^{n-d} \delta_i(V).$$

Proof. Consider the polynomials $P_1, \dots, P_{n-d}$ of the type provided by Lemma 3.2.6 associated to V and let $Z_1, \dots, Z_r$ be the irreducible components of $Z(P_1, \dots, P_{n-d})$. We know that V is an irreducible component of dimension d of this algebraic set. Therefore, by Lemma 3.1.5 we find the following bound:

$$\deg(V) \leq \sum_{i=1}^{r} \deg(Z_i) \leq \prod_{i=1}^{n-d} \deg(P_i) = \prod_{i=1}^{n-d} \delta_i(V).$$

□

Definition 3.2.9 (Admissible integer)**.** We say a non-negative integer i is *admissible with respect to* V if $\delta_{i+1}(V) > 2i\delta_i(V)$.

Notice that if i is not admissible, then $\delta_i(V) \gtrsim_n \delta_{i+1}(V)$. If we let $c_1 \gtrsim_n 1$ be a sufficiently small constant, then for every irreducible variety $V \subseteq \mathbb{C}^n$ we have that every positive integer lies inside an interval of the form

$$\mathcal{R}_i(V) = [c_1 \Pi_i(V), \frac{c_0}{2}\delta_{i+1}(V)^{n-i}\Delta_i(V)],$$

where c_0 is the constant given in Theorem 3.3.3. We can extend the definition of these intervals as follows. Suppose that $V \subseteq \mathbb{C}^n$ is an irreducible algebraic variety of dimension d, then for every $0 \leq s \leq n-d$ choose a real number $\tau > 0$ and for every $0 \leq l \leq n-s$ define the interval

$$\mathcal{R}^l_{s,\tau}(V) = [\tau\delta_s(V)^{n-(s+l)}\Delta_s(V), \tau\delta_{s+1}(V)^{n-(s+l)}\Delta_s(V)].$$

We have the following observation.

Lemma 3.2.10. *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety. For any integers $l < d$ and $0 < \epsilon < 1$, we can find $\epsilon \lesssim_n \tau_1, \dots, \tau_{n-d} \leq \epsilon$ such that $\mathbb{R}_{\geq 0}$ is covered by the sets $\mathcal{R}^l_{s,\tau_s}(V)$ with $\mathcal{S}$ admissible.*

Theorem 3.2.11. *Let $V \subseteq \mathbb{C}^n$ be an irreducible variety of dimension d and let $P_1, \dots, P_{n-d}$ be the corresponding polynomials given by lemma Lemma 3.2.6. Then, for every $1 \leq m \leq n-d$, every irreducible component W of $Z(P_1, \dots, P_m)$ having dimension $n-m$ and containing V has degree $\sim_n \prod_{i=1}^m \delta_i(V)$ and satisfies $\delta_i(W) \sim_n \delta_i(V)$ for every $1 \leq i \leq m$.*

Proof. We proceed by induction on m. Suppose that $m = 1$. We have $\dim(Z(P_1)) = n-1$ and $\deg(Z(P_1)) = \deg(P_1) = \delta_1(V)$. Moreover, by definition of partial degree we know that $\delta_1(Z(P_1)) = \deg(P_1)$ and we are done with the base case.
Let $m > 1$ and consider the result true for every smaller value of m. Since W is an irreducible component of $Z(P_1, \dots, P_m)$ we can use Lemma 3.1.5 to obtain the following bound.

$$\deg(W) \leq \prod_{i=1}^{m} \deg(P_i) = \prod_{i=1}^{m} \delta_i(V). \tag{3.2.1}$$

Let us now proceed by contradiction. Assume that for any sufficiently small constant $\epsilon \gtrsim_n 1$ with respect to n we have the inequality

$$\deg(W) \leq \epsilon \prod_{i=1}^{m} \delta_i(V). \tag{3.2.2}$$

Let $Z_1, \ldots, Z_r$ be the irreducible components of $Z(P_1, \ldots, P_{m-1})$ of dimension $n-(m-1)$ containing V. By Lemma 3.2.10 there exists some τ_{n-m} such that $\deg(W) \in \mathcal{R}^{n-m}_{s,\tau_{n-m}}$ for any admissible integer $\mathcal{S}$. Then, for every $1 \leq j \leq r$ we can use Theorem 4.0.3 to find a polynomial Q_j of degree at most

$$\lesssim_n \left(\frac{\deg(W)}{\Delta_{s_j}(Z_j)}\right)^{\frac{1}{m-s_j}},$$

vanishing on W without vanishing on Z_j for some admissible integer $0 \leq s_j < m$.

Claim 3.2.12. $\Delta_{s_j}(Z_j) \sim_n \prod_{i=1}^{s_j} \delta_i(V)$ *for every* $1 \leq j \leq r$.

Proof. Observe that the induction hypothesis can be applied to the components Z_j. This means that $\deg(Z_j) \sim_n \delta_1(V) \cdots \delta_{m-1}(V)$ for every $1 \leq j \leq r$ and $\delta_i(Z_j) \sim_n \delta_i(V)$ for every $1 \leq i \leq m-1$ and $1 \leq j \leq r$. Using the definition of $\Delta_{s_j}(Z_j)$ we finish the proof. ☐

Applying the claim together with the bound (3.2.2) to the degree of the polynomials we have found we obtain

$$\begin{aligned}\deg(Q_j) \lesssim_n \left(\frac{\epsilon \prod_{i=1}^{m} \delta_i(V)}{\prod_{i=1}^{s_j} \delta_i(V)}\right)^{\frac{1}{m-s_j}} &= (\epsilon \prod_{i=s_j+1}^{m} \delta_i(V))^{\frac{1}{m-s_j}} \\ \leq (\epsilon \delta_m(V)^{m-s_j})^{\frac{1}{m-s_j}} &= \epsilon^{\frac{1}{m-s_j}} \delta_m(V).\end{aligned}$$

Notice that $\epsilon^{\frac{1}{m-s_j}} \delta_m(V) < \delta_m(V)$ if ϵ is sufficiently small. Now, by Lemma 3.2.5 we find a polynomial Q vanishing on W without vanishing at any of the components Z_j. This means that $V \subseteq W \subseteq Z(P_1, \ldots, P_{m-1}, Q)$. Moreover, since Q does not cut every component Z_j properly each of the irreducible components of $Z(P_1, \ldots, P_{m-1}, Q)$ containing V has dimension at most $n-m$. Since P_m is a polynomial of the smallest possible degree such that the maximal dimension of an irreducible component of $Z(P_1, \ldots, P_m)$ containing V has degree at most $n-m$ and $\deg(Q) < \delta_m(V) = \deg(P_m)$ we have found a contraduction. We conclude that

$$\deg(W) \sim_n \prod_{i=1}^{m} \delta_i(V). \tag{3.2.3}$$

To finish the proof we need to see that $\delta_i(W) \sim_n \delta_i(V)$ for every $1 \leq i \leq m$. We know that for every $1 \leq i \leq m$ we have $V \subseteq W \subseteq Z(P_1, \ldots, P_i)$. This implies that

$\delta_i(W) \leq \delta_i(V)$ for every i. By Corollary 3.2.8 we know that $\deg(W) \leq \prod_{i=1}^m \delta_i(W)$. Moreover, we have just seen that there exists some constant c depending on n such that $c\prod_{i=1}^m \delta_i(V) \leq \deg(W)$. Thus we get

$$c\prod_{i=1}^{m} \delta_i(V) \leq \prod_{i=1}^{m} \delta_i(W) \implies c\delta_i(V) \leq \delta_i(W) \text{ for every } 1 \leq i \leq m,$$

as we wanted to see. □

Corollary 3.2.13. *Consider an irreducible variety $V \subseteq \mathbb{C}^n$ of dimension d and let W be an irreducible variety of dimension $n-k$ containing V. Then $\deg(W) \gtrsim_n \delta_1(V)\cdots\delta_k(V)$.*

Proof. By definition of partial degree we know that, for every $k+1 \leq s \leq n-d$ we can find a set of polynomials $g_{s_1}, \ldots, g_{s_t} \in \mathbb{C}[x_1, \ldots, x_n]$ such that each of them vanishing on V without vanishing on W. Applying Lemma 3.2.5 to each set of the previous polynomials we obtain a collection of $n-d-k$ polynomials $f_{k+1}, \ldots, f_{n-d}$ such that for every s we have $\deg(f_s) \leq \delta_s(V)$ and the maximal dimension of an irreducible component of $W \cap Z(f_{k+1}, \ldots, f_s)$ containing V is equal to $(n-k)-(s-k) = n-s$. Taking $s = n-d$ we see that V is an irreducible component of $W \cap Z(f_{k+1}, \ldots, f_{n-d})$. Applying Theorem 3.2.11 we find that

$$\deg(V) \sim_n \prod_{i=1}^{n-d} \delta_i(V).$$

Moreover, by Lemma 3.1.5 we know that

$$\deg(W \cap Z(f_{k+1}, \ldots, f_{n-d})) \leq \deg(W) \prod_{i=k+1}^{n-d} \deg(f_i) \leq \deg(W) \prod_{i=k+1}^{n-d} \delta_i(V).$$

Since V is an irreducible component of $W \cap Z(f_{k+1}, \ldots, f_{n-d})$ we must have

$$\deg(V) \leq \deg(W \cap Z(f_{k+1}, \ldots, f_{n-d})).$$

Therefore, for some constant c depending on n we obtain:

$$c\prod_{i=1}^{n-d} \delta_i(V) \leq \deg(W) \prod_{i=k+1}^{n-d} \delta_i(V) \implies \deg(W) \geq c\prod_{i=1}^{k} \delta_i(V),$$

as we wanted to see. □

3.3 Hilbert functions

We finish this chapter by stating a few results concerning *Hilbert functions*. They will be needed in chapter 4 when proving the generalisation of Siegel's lemma.

Notation 3.3.1. Let $V \subseteq \mathbb{C}^n$ be an algebraic set. We write $I(V)_{\leq m}$ for those elements of $I(V)$ of degree at most m. We also write $I_{\mathbb{R}}(V)$ for the ideal of $\mathbb{R}[x_1, \dots, x_n]$ consisting of the real polynomials of $I(V)$.

Definition 3.3.2 (Hilbert function)**.** Consider an algebraic set $V \subseteq \mathbb{C}^n$. To the ideal $I(V)$ of V we can associate the *affine Hilbert function*

$$H_{I(V)}(m) := \dim_{\mathbb{C}}(\mathbb{C}[x_1, \dots, x_n]_{\leq m}/I(V)_{\leq m}).$$

We can also define the function

$$H_{I(V),\mathbb{R}}(m) := \dim_{\mathbb{R}}(\mathbb{R}[x_1, \dots, x_n]_{\leq m}/I_{\mathbb{R}}(V)_{\leq m}).$$

Theorem 3.3.3. *Let $V \subseteq \mathbb{C}^n$ be an algebraic set having all its irreducible components of dimension d. Then, there exists some constant $c_0 \gtrsim_n 1$ such that, for every $m \geq 2(n-d)\delta(V)$, we have the bound*

$$H_{I(V)}(m) \geq c_0 m^d \deg(V).$$

Lemma 3.3.4. *We have $H_{I(V),\mathbb{R}}(m) \geq H_{I(V)}(m)$ for every algebraic variety $V \subseteq \mathbb{C}^n$ and every $m \geq 0$.*

Lemma 3.3.5. *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d. Let s be an admissible integer with respect to V and $k \in \mathcal{R}_s(V)$. Let m be a positive integer such that $2s\delta_s(V) < m < \delta_{s+1}(V)$ and $c_0 m^{n-s}\Delta_s(V) > k$. Then $H_{I(V)}(m) > k$.*

Proof. Let $g_1, \dots, g_t$ be polynomials satisfying the conditions of the definition of $\delta_s(V)$. Let $V_1, \dots, V_r$ be the irreducible components of $Z(g_1, \dots, g_t)$ of dimension $n-s$ containing V.

Claim 3.3.6. *There exists some $1 \leq j \leq r$ such that $I(V)_{\leq m} = I(V_j)_{\leq m}$.*

Proof. Since $V \subseteq V_j$ for all $1 \leq j \leq r$ then $I(V_j)_{\leq m} \subseteq I(V)_{\leq m}$ for all $1 \leq j \leq r$. To see the other inclusion suppose that for every $1 \leq j \leq r$ there is some polynomial $h_j \in I(V)_{\leq m} \setminus I(V_j)_{\leq m}$. Then $V_j \cap Z(h_j)$ is an algebraic set which contains V and having all its irreducible components of dimension less than $n-s$. Therefore the zero set $Z(g_1, \dots, g_t, h_1, \dots, h_r)$ is an algebraic set containing V and its irreducible components that contain V have dimension at most $n-s-1$. Since all the polynomials $g_1, \dots, g_t, h_1, \dots, h_r$ have degree at most m, by definition of partial degree we obtain that $\delta_{s+1}(V) \leq m$ which is a contradiction. ☐

Without loss of generality we can assume that $I(V_1)_{\leq m} = I(V)_{\leq m}$. Since V_1 is irreducible of dimension $n-s$ and $m > 2s\delta_s(V)$ we can use Theorem 3.3.3 to conclude that

$$H_{I(V)}(m) = H_{I(V_1)}(m) \geq c_0 m^{n-s} \deg(V_1). \tag{3.3.1}$$

Claim 3.3.7. $\deg(V_1) \geq \Delta_s(V)$.

Proof. By definition of $\delta_{s+1}(V)$ we can find polynomials $g_1, \ldots, g_t \in \mathbb{C}[x_1, \ldots, x_n]$ of degree at most $\delta_{s+1}(V)$ such that $V \subseteq Z(g_1, \ldots, g_t)$ and the irreducible components of $Z(g_1, \ldots, g_t)$ containing V have dimension at most $n - s - 1$. Since $V \subseteq V_1$ and $\dim(V_1) = n - s$ it must be that $V_1 \not\subseteq Z(g_1, \ldots, g_t)$, otherwise we would contradict the definition of partial degree. Let f_{s+1} be any of the polynomials $g_1, \ldots, g_s$. Then f_{s+1} is a polynomial of degree at most $\delta_{s+1}(V)$ vanishing on V without vanishing on V_1. This means that there exists an irreducible component W_{s+1} of $Z(f_{s+1}) \cap V_1$ containing V of dimension $n - s - 1$. Moreover, by bezout's inequality Lemma 3.1.5 we know that

$$\deg(W_{s+1}) \leq \deg(V_1) \deg(f_{s+1}) \leq \deg(V_1)\delta_{s+1}(V).$$

We can now proceed by applying the same reasoning to the variety W_{s+1}. By definition of $\delta_{s+2}(V)$ we can find a polynomial f_{s+2} of degree at most $\delta_{s+2}(V)$ such that $Z(f_{s+2}) \cap W_{s+1}$ contains an irreducible variety W_{s+2} containing V of dimension $n - s - 2$ and of degree $\leq \deg(W_{s+1}) \deg(f_{s+2}) \leq \deg(V_1)\delta_{s+1}(V)\delta_{s+2}(V)$. Iterating this argument we find an irreducible variety W_{n-d} such that

$$\deg(W_{n-d}) \leq \deg(V_1)\delta_{s+1}(V) \cdots \delta_{n-d}(V).$$

Since $\dim(W_{n-d}) = d$ it must be that $V = W_{n-d}$. □

By hypothesis and Claim 3.3.7 we conclude that

$$H_{I(V)}(m) \geq c_0 m^{n-s} \Delta_s(V) > k.$$

□

Chapter 4

Siegel's lemma for varieties

The aim of this chapter is to prove Theorem 1.0.13 and Theorem 4.0.3 which have already been used in chapter 3 and we will use them again in chapter 6.
Since both theorems have very similar proofs we will give the proof of Theorem 4.0.3 which is more general and then indicate the necessary changes to prove Theorem 1.0.13. Since the proofs are done by induction we will deal with the base cases first proving Lemma 4.0.1 and Lemma 4.0.5. The material for this chapter has been extracted from [Wal18, Section 4].

Lemma 4.0.1. *Let $\mathcal{S}$ be a finite subset of $\mathbb{C}^n$ and let V be a d-dimensional irreducible variety $V \subseteq \mathbb{C}^n$. Let $\tau > 0$ be sufficiently small with respect to n and let s be an admissible integer with $|\mathcal{S}| \in \mathcal{R}^0_{s,\tau}(V)$. Then, there exists some polynomial P of degree at most*

$$\lesssim_{n,\tau} \left(\frac{|\mathcal{S}|}{\Delta_s(V)}\right)^{\frac{1}{n-s}},$$

vanishing on $\mathcal{S}$ without vanishing identically on V.

Proof. Let $p_1, \ldots, p_t$ be a basis of $\mathbb{C}[x_1, \ldots, x_n]_{\leq m}/I(V)_{\leq m}$. We start by proving the following claim.

Claim 4.0.2. $t > |\mathcal{S}|$.

Proof. Since $|\mathcal{S}| \in \mathcal{R}^0_{s,\tau}(V)$ we have

$$2s\delta_s(V) \lesssim_\tau \left(\frac{|\mathcal{S}|}{\Delta_s(V)}\right)^{\frac{1}{n-s}} \lesssim_\tau \delta_{s+1}(V)$$

Therefore by definition of admissible integer we can find $2s\delta_s(V) < m < \delta_{s+1}(V)$ such that

$$m \lesssim_{n,\tau} \left(\frac{|\mathcal{S}|}{\Delta_s(V)}\right)^{\frac{1}{n-s}} \text{ and } |\mathcal{S}| < c_0 m^{n-s}\Delta_s(V).$$

Therefore by Lemma 3.3.5 we know that

$$t = H_{I(V)}(m) > |\mathcal{S}|.$$

□

Consider elements $q_1, \dots, q_t$ of $\mathbb{C}[x_1, \dots, x_n]_{\leq m}$ such that $\pi(q_i) = p_i$ for every i where π is the projection

$$\pi : \mathbb{C}[x_1, \dots, x_n]_{\leq m} \to \mathbb{C}[x_1, \dots, x_n]_{\leq m}/I(V)_{\leq m}.$$

Since $t > |\mathcal{S}|$ there exists some linear combination $q = \sum_{i=1}^{t} c_i q_i$ with $c_i \in \mathbb{C}$ for every i such that $q \in I(\mathcal{S})$. Moreover, since $\pi(q) = \sum_{i=1}^{t} c_i p_i \notin I(V)$ we obtain that $q \notin I(V)$. □

Theorem 4.0.3 (Siegel's lemma for varieties)**.** *Let $0 \leq l < d \leq n$ be integers and $\tau_l > 0$ a sufficiently small constant with respect to n. Let T be a finite set of l-dimensional irreducible algebraic varieties in $\mathbb{C}^n$ and V a d-dimensional irreducible algebraic variety in $\mathbb{C}^n$. Let $0 \leq s \leq n-d$ be an admissible integer with $\deg(T) \in \mathcal{R}^l_{s,\tau_l}(V)$. Then, there exists some polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ of degree at most*

$$\lesssim_{n,\tau_l} \left(\frac{\deg(T)}{\Delta_s(V)} \right)^{\frac{1}{n-(s+l)}},$$

vanishing at all elements of T without vanishing identically on V.

Proof. We will proceed by induction on l. It is clear that the case $l = 0$ corresponds to Lemma 4.0.1. First notice that since $\deg(T) \in \mathcal{R}^l_{s,\tau_l}(V)$ we have that

$$\tau_l \delta_s(V)^{n-(s+l)} \Delta_s(V) \leq \deg(T) \leq \tau_l \delta_{s+1}^{n-(s+l)}(V) \Delta_s(V),$$

in particular we find that

$$\tau_l^{\frac{1}{n-(s+l)}} \delta_s(V) \leq \left(\frac{\deg(T)}{\Delta_s(V)} \right)^{\frac{1}{n-(s+l)}} \leq \tau_l^{\frac{1}{n-(s+l)}} \delta_{s+1}(V). \tag{4.0.1}$$

Let $R \geq 1$ be the following parameter

$$R = B \left(\frac{\deg(T)}{\Delta_s(V)} \right)^{\frac{1}{n-(s+l)}}$$

for some large $B \sim_{n,\tau_l} 1$. Pick any irreducible variety $h \subseteq \mathbb{C}^n$ of dimension $l-1$ and some $t \in T$. Then by Lemma 3.1.9 there exists some irreducible variety $h_1 \subset t$ of dimension $l-1$ with $\deg(h_1) \leq \deg(t)$. Taking $H = \{h_1\}$ and applying Lemma 3.1.9 again we find another subvariety $h_2 \subseteq t$ such that $\deg(h_2) \leq \deg(t)$ and $h_2 \notin H$. Repeating this process we can find a collection of $r \geq 1$ distinct subvarieties

$h_1, \dots, h_r$ of t of dimension $l-1$ such that $\deg(h_i) \le \deg(t)$ for every $1 \le i \le r$. This means that we have the inequality

$$\sum_{i=1}^{r} \deg(h_i) \le r \deg(t).$$

Observe that we can choose r in such a way that the following inequalities hold:

$$2R \deg(t) \le \sum_{i=1}^{r} \deg(h_i) \le (2R+1) \deg(t).$$

For any other element $t_1 \in T$ we can set $H = \{h_1, \dots, h_r\}$ and apply Lemma 3.1.9, which yields a collection of distinct subvarieties of t_1 giving the same bounds as above for some suitable r. If we repeat this process for every remaining element of T, always ensuring that the subvarieties are distinct between each other and the previous ones, we find a collection of varieties $\mathcal{H}$ of dimension $l-1$ such that

$$2R \deg(T) \le \deg(\mathcal{H}) \le (2R+1) \deg(T). \tag{4.0.2}$$

With this bound and our choice of R we obtain that

$$\deg(\mathcal{H}) = C_1 B \frac{\deg(T)^{1+\frac{1}{n-(s+l)}}}{\Delta_s(V)^{\frac{1}{n-(s+l)}}},$$

with $C_1 \sim 1$. Moreover, using the inequalities in (4.0.1) it follows that

$$\tau_l^{\frac{1}{n-(s+l)}} \delta_s(V) \le \frac{\deg(\mathcal{H})}{C_1 B \deg(T)} \le \tau_l^{\frac{1}{n-(s+l)}} \delta_{s+1}(V). \tag{4.0.3}$$

Using that $\deg(T) \in \mathcal{R}^l_{s,\tau_l}$ we obtain the following bounds.

$$(\tau_l^{1+\frac{1}{n-(s+l)}} C_1 B) \delta_s(V)^{n-(s+l-1)} \Delta_s(V) \le \deg(\mathcal{H}) \le (\tau_l^{1+\frac{1}{n-(s+l)}} C_1 B) \delta_{s+1}(V)^{n-(s+l-1)} \Delta_s(V) \tag{4.0.4}$$

Let $\tau_{l-1} \sim_n 1$ be a sufficiently small constant and let $B_0 \sim_n 1$ be a sufficiently large constant with respect to τ_{l-1} and n. We now choose τ_l sufficiently small in order to get the inequality

$$\tau_l^{1+\frac{1}{n-(s-l)}} C_1 B_0 \le \tau_{l-1}.$$

We finish by setting $B \ge B_0$ in such a way that the following holds:

$$\tau_l^{1+\frac{1}{n-(s-l)}} C_1 B = \tau_{l-1}.$$

Therefore by (4.0.4) we get that $\deg \mathcal{H} \in \mathcal{R}^{l-1}_{s,\tau_{l-1}}(V)$. We can now apply the induction hypotesis. We find a polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ of degree at most

$$\lesssim_{n,\tau_{l-1}} \left(\frac{\deg(\mathcal{H})}{\Delta_s(V)} \right)^{\frac{1}{n-(s+l-1)}},$$

vanishing at all elements of $\mathcal{H}$ without vanishing identically on V. More precisely, this means that

$$\deg(P) \lesssim_{n,\tau_{l-1}} B^{\frac{1}{n-(s+l-1)}} \left(\frac{\deg(T)}{\Delta_s(V)}\right)^{\frac{1}{n-(s-l)}}.$$

Claim 4.0.4. $\deg(P) < R$

Proof. We took $B \geq B_0$ and we can freely choose $B_0 \sim_n 1$. Moreover, since $s \leq n-d$ and $l < d$ we know that $n-(s+l-1) \geq 2$. Therefore $B^{\frac{1}{n-(s+l-1)}} < B$ and since $\tau_{l-1} \lesssim_n 1$ we get the result. □

Suppose that P does not vanish at some $t \in T$. Then by Lemma 3.1.5 we know that $\deg(Z(P) \cap t) \leq \deg(P)\deg(t) < R\deg(T)$. However, since $\mathcal{H} \subseteq Z(P)$ we know that $Z(P) \cap t$ contains the components comming from the intersection $\mathcal{H} \cap t$. However the sum of these irreducible components have degree at least $2R\deg(t)$ giving a contradiction. Therefore P is a polynomial which vanishes at all elements of T without vanishing identically on the variety V. Since $B \sim_{n,\tau_l}$ and $\deg(P) < R$ as we wanted to see. □

We now turn our attention to the proof of Theorem 1.0.13. Let us start by the base case for the induction.

Lemma 4.0.5. *Let $V \subseteq \mathbb{C}^n$ be a d-dimensional algebraic set and $\tau > 0$ some real number. Let $\mathcal{S}$ be a finite subset of $\mathbb{C}^2$ with $|\mathcal{S}| \geq \tau\delta(V)^d\deg(V)$. Then, there exists some polynomial P of degree at most*

$$\lesssim_{n,\tau} \left(\frac{|\mathcal{S}|}{\deg(V)}\right)^{1/d},$$

vanishing on $\mathcal{S}$ without vanishing identically on V.

Proof. Let $p_1, \ldots, p_t$ be a basis of $\mathbb{C}[x_1, \ldots, x_n]_{\leq m}/I(V)_{\leq m}$. As we did with Lemma 4.0.1, we start by proving the following claim.

Claim 4.0.6. $t > |\mathcal{S}|$.

Proof. Since $|\mathcal{S}| \geq \tau\delta(V)^d\deg(V)$ we immediately get the bound

$$\delta(V) \lesssim_n \left(\frac{|\mathcal{S}|}{\deg(V)}\right)^{1/d}.$$

Pick an integer m such that

$$m \geq 2(n-d)\delta(V) \gtrsim_n 2(n-d)\left(\frac{|\mathcal{S}|}{\deg(V)}\right)^{1/d}.$$

Then, since V is a d-dimensional algebraic set we can use Theorem 3.3.3 to obtain the bound

$$H_{I(V)}(m) \geq c_0 m^d \deg(V) \gtrsim_n c_0 2^d (n-d)^d |\mathcal{S}| > |\mathcal{S}|.$$

□

The rest of the argument is the same as in the proof of Lemma 4.0.1. □

Proof of Theorem 1.0.13. We proceed by induction on l. The base case corresponds to *Lemma* 4.0.5. Let $R \geq 1$. We use the same reasoning as in the proof of Theorem 4.0.3 to obtain a collection $\mathcal{H}$ of subvarieties such that

$$2R\deg(T) \leq \deg(\mathcal{H}) \leq (2R+1)\deg(T).$$

In this case we can take $s = n-d$ which implies that $\Delta_{n-d}(V) = \deg(V)$. Therefore, we can set

$$R = B\left(\frac{\deg(T)}{\deg(V)}\right)^{\frac{1}{d-l}}$$

for some $B \sim_{n,\tau_l} 1$. As we did in the proof of Theorem 4.0.3, this means that

$$\deg(\mathcal{H}) = C_1 B \frac{\deg(T)^{1+\frac{1}{d-l}}}{\deg(V)^{\frac{1}{d-l}}},$$

with $C_1 \sim 1$. Since $\deg(T) \geq \tau_l \delta(V)^{d-l} \deg(V)$, we obtain the following bound:

$$\deg(\mathcal{H}) \geq C_1 B \tau_l^{1+\frac{1}{d-l}} \delta(V)^{d-l+1} \deg(V).$$

We can choose τ_{l-1} and B_0 as we did in the proof of Theorem 4.0.3 to conclude that

$$\tau_l^{1+\frac{1}{d-l}} C_1 B = \tau_{l-1}.$$

We are now under the induction hypothesis and we can finish the proof arguing analogously as we did for Theorem 4.0.3. □

Chapter 5

Polynomial partitioning theorem for varieties

In chapter 2 we saw how to prove the Guth-Katz polynomial partitioning theorem over $\mathbb{R}^2$ (Theorem 2.2.5) using a polynomial version (Theorem 2.2.4) of the ham sandwich theorem of Stone and Tukey. The strategy of the proof was to recursively use Theorem 2.2.4 to produce algebraic surfaces that simultaneously bisect a collection of finite sets of points in $\mathbb{R}^2$.

The goal of this chapter is to prove a polynomial partitioning theorem for generic algebraic varieties. The strategy of the proof is the same as the one described above. The material for this chapter was obtained from [Wal18, Section 3].

5.1 Polynomial ham sandwich theorem

Theorem 5.1.1 (Ham sandwich theorem for varieties)**.** *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d and let $\mathcal{S}_1, \ldots, \mathcal{S}_k$ be finite subsets of $V(\mathbb{R})$. Let s be an admissible integer with respect to V such that $k \in \mathcal{R}_s(V)$. Then there exists a real polynomial $g \notin I(V)$ of degree at most*

$$\lesssim_n \left(\frac{k}{\Delta_s(V)}\right)^{\frac{1}{n-s}}$$

that bisects every $\mathcal{S}_i$.

Proof. Since

$$c_1\delta_s(V)^{n-s}\Delta_s(V) \leq k \leq \frac{c_0}{2}\delta_{s+1}(V)^{n-s}\Delta_s(V)$$

and s is an admissible integer with respect to V, we can find some positive integer m such that $2s\delta_s(V) < m < \delta_{s+1}(V)$ and

$$m \lesssim_n \left(\frac{k}{\Delta_s(V)}\right)^{\frac{1}{n-s}} \quad \text{and} \quad c_0m^{n-s}\Delta_s(V) > k,$$

where c_0 is as in Theorem 3.3.3. Thus, it suffices to prove that there exists some real polynomial $g \notin I(V)$ of degree at most m bisecting every set $\mathcal{S}_i$.

Let $1, p_1, \ldots, p_t$ be a basis of $\mathbb{R}[x_1, \ldots, x_n]_{\leq m}/I_{\mathbb{R}}(V)_{\leq m}$. By Lemma 3.3.5 and Lemma 3.3.4 we have that $t \geq k$. To each p_i we associate an element $q_i \in \mathbb{R}[x_1, \ldots, x_n]_{\leq m}$ whose projection to $\mathbb{R}[x_1, \ldots, x_n]_{\leq m}/I_{\mathbb{R}}(V)_{\leq m}$ is equal to p_i. Consider the map

$$\phi : \mathbb{R}^n \to \mathbb{R}^t \text{ given by } \phi(x) = (q_1(x), \ldots, q_t(x)).$$

Claim 5.1.2. *The map ϕ defined above is injective on $V(\mathbb{R})$.*

Proof. Let $x, y \in V(\mathbb{R})$ such that $x \neq y$. We know that there is some $1 \leq i \leq n$ such that the linear projection π_i to the *ith* coordinate satisfies $\pi_i(x) \neq \pi_i(y)$. Since the elements of $I(V)$ vanish on both x and y, and $1, p_1, \ldots, p_t$ is a basis for $\mathbb{R}[x_1, \ldots, x_n]_{\leq m}/I_{\mathbb{R}}(V)_{\leq m}$ there must exists some linear combination of the q_i that takes different values on x and y. This proves the claim. □

Consider the sets $\phi(\mathcal{S}_1), \ldots, \phi(\mathcal{S}_k) \in \mathbb{R}^t$. By Theorem 5.1.1 and the fact that $k \leq t$, we know that there exists some hyperplane in $\mathbb{R}^t$ bisecting each $\phi(\mathcal{S}_i)$. That is, there exist some coefficients $a_1, \ldots, a_{t+1} \in \mathbb{R}$ not all zero such that for every $\mathcal{S}_i$ we have

$$|\{x \in \mathcal{S}_i \ : \ a_1 q_1(x) + \cdots + a_t q_t(x) + a_{t+1} > 0\}| \leq |\phi(\mathcal{S}_i)|/2 = |\mathcal{S}_i|/2,$$

$$|\{x \in \mathcal{S}_i \ : \ a_1 q_1(x) + \cdots + a_t q_t(x) + a_{t+1} < 0\}| \leq |\phi(\mathcal{S}_i)|/2 = |\mathcal{S}_i|/2.$$

The polynomial that we were looking for is $g = a_1 q_1 + \cdots + a_t q_t + a_{t+1}$. □

5.2 Polynomial partitioning

Notation 5.2.1. Given an irreducible algebraic variety V and an integer M, we write $i_V(M)$ for the smallest admissible integer i such that $M^{n-i}\Delta_i(V) \in \mathcal{R}_i(V)$.

Theorem 5.2.2 (Polynomial partitioning for varieties)**.** *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d and $\mathcal{S}$ a finite set of points inside of $V(\mathbb{R})$. Then, given any integer $M \geq 1$, we can find some polynomial $P \in \mathbb{R}[x_1, \ldots, x_n] \setminus I(V)$ of degree $O_n(M)$ such that each connected component of $\mathbb{R}^n \setminus Z(P)$ contains*

$$\lesssim_n \frac{|\mathcal{S}|}{M^{n-i_V(M)}\Delta_{i_V(M)}(V)}$$

elements of $\mathcal{S}$.

Proof. We are going to divide the proof in two parts. First, we will show how to construct the polynomial P leaving the proof regarding its degree for the end.

Since we can find an admissible integer s_1 with respect to V such that $1 \in \mathcal{R}_{s_1}(V)$

it follows from Theorem 5.1.1 that there exists a polynomial $h_1 \notin I(V)$ of degree $O(1)$ bisecting $\mathcal{S}$. Let us write

$$A_{1,1} := \{x \in \mathbb{R}^n \mid h_1(x) > 0\}, \quad A_{1,2} := \{x \in \mathbb{R}^n \mid h_1(x) < 0\},$$

$$S_{1,1} := A_{1,1} \cap S \quad \text{and} \quad S_{1,2} = A_{1,2} \cap S.$$

Notice that each set $A_{1,1}$ and $A_{1,2}$ is the union of some open connected components of $\mathbb{R}^n \setminus Z(h_1)$. Moreover, since h_1 bisects $\mathcal{S}$ we must have $|S_{1,1}| \leq \frac{|\mathcal{S}|}{2}$, $|S_{1,2}| \leq \frac{|\mathcal{S}|}{2}$ and $S \setminus (S_{1,1} \cup S_{1,2}) \subseteq Z(g_1)$.

There is an admissible integer s_2 with respect to V such that $2 \in \mathcal{R}_{s_2}(V)$, thus we can apply Theorem 5.1.1 again on the sets $S_{1,1}$ and $S_{1,2}$ finding a polynomial $h_2 \notin I(V)$ bisecting them. Consider the polynomial $g_2 := h_1 h_2$. By construction of h_1 and h_2 it is clear that $g_2 \notin I(V)$. Now we want to split the points of $A_{1,1}$ and $A_{1,2}$ according to the sign of h_2. Consider the sets

$$B_{2,1} := \{x \in \mathbb{R}^n \mid h_2(x) > 0\} \text{ and } B_{2,2} := \{x \in \mathbb{R}^n \mid h_2(x) < 0\}$$

and define the sets

$$A_{2,1} = A_{1,1} \cap B_{2,1}, \quad A_{2,2} = A_{1,2} \cap B_{2,1}, \quad A_{2,3} = A_{1,1} \cap B_{2,2} \text{ and } A_{2,4} = A_{1,2} \cap B_{2,2}.$$

These open sets are the union of some open connected components of $\mathbb{R}^n \setminus Z(g_2)$. Now define the sets

$$S_{2,1} = A_{2,1} \cap S, \quad S_{2,2} = A_{2,2} \cap S, \quad S_{2,3} = A_{2,3} \cap S \text{ and } S_{2,4} = A_{2,4} \cap S.$$

Since h_2 bisects $S_{1,1}$ and $S_{1,2}$ they have at most half of their elements in $B_{2,1}$ and half of them in $B_{2,2}$. Moreover $S_{1,1} \subset A_{1,1}$ and $S_{1,2} \subset A_{1,2}$ therefore we can conclude that $|S_{2,i}| \leq \frac{|\mathcal{S}|}{4}$ for $1 \leq i \leq 4$.

Using the same process as illustrated above for every positive integer $i \leq \log_2 r$ with $r = M^{n-i_V(M)}\Delta_{i_V(M)}(V)$ we can suppose that we have constructed a real polynomial $g_{i-1} \notin I(V)$ and disjoint sets $A_{i-1,1}, \ldots, A_{i-1,2^{i-1}}$ each of them being the union of some open connected components of $\mathbb{R}^n \setminus Z(g_{i-1})$. Moreover, we have also the sets $S_{i-1,j}$ which correspond to the points of $\mathcal{S}$ inside of $A_{i-1,j}$ and by construction we have $|S_{i-1,j}| \leq |\mathcal{S}|2^{-(i-1)}$. Since there is an admissible integer s_{i-1} with respect to V such that $2^{i-1} \in \mathcal{R}_{s_{2^{i-1}}}(V)$ we can apply Theorem 5.1.1 to find a real polynomial $h_i \notin I(V)$ bisecting $S_{i-1,j}$ for every $1 \leq j \leq 2^{i-1}$. Let us write $g_i = g_{i-1}h_i$ and notice that $g_i \notin I(V)$ and g_i is a real polynomial. Let

$$B_{i,1} = \{x \in \mathbb{R}^n \mid h_i(x) > 0\}, \quad B_{i,2} = \{x \in \mathbb{R} \mid h_i(x) < 0\}.$$

These are open sets with boundary in $Z(h_i)$. For every $1 \leq j \leq 2^{i-1}$ define

$$A_{i,j} = A_{i-1,j} \cap B_1 \text{ and } A_{i,2^{i-1}+j} = A_{i-1,j} \cap B_2.$$

These sets are open sets which are the union of some open connected components of $\mathbb{R}^n \setminus Z(g_i)$. Consider now the sets

$$S_{i,j} = A_{i,j} \cap S.$$

They form a collection of 2^i sets, with each $|S_{i,j}| \leq |\mathcal{S}|2^{-i}$ elements of $\mathcal{S}$. All elements of $\mathcal{S}$ not lying inside of $S_{i,j}$ for any j must lie inside of $Z(g_i)$. Therefore the polynomial that we are looking for is

$$P := g_{\log_2 r} = \prod_{i=1}^{\log_2 r} h_i.$$

For the second part of the proof, let us show how to bound the degree of P. First notice that we have

$$\Delta_{i+1}(V) \leq \delta_{i+1}(V)\Delta_i(V).$$

It follows that

$$\Pi_{t+1}(V) = \delta_{t+1}(V)^{n-t-1}\Delta_{t+1}(V) \leq \delta_{t+1}(V)^{n-t-1}\delta_{t+1}(V)\Delta_t(V) = \delta_{t+1}(V)^{n-t}\Delta_t(V).$$

Hence we obtain the inclusion

$$[c_1\Pi_t(V), \frac{c_0}{2}\Pi_{t+1}(V)] \subseteq [c_1\Pi_t(V), \frac{c_0}{2}\delta_{t+1}(V)^{n-t}\Delta_t(V)] = \mathcal{R}_t.$$

If we let t be an admissible integer with respect to V, these intervals cover the positive integers. If $t < i_V(M)$ is the smallest admissible integer with 2^{i-1} lying in an interval of the above form, we can use Theorem 5.1.1 to obtain the bound:

$$\deg(h_i) \lesssim_n 2^{\frac{i-1}{n-t}}\Delta_t(V)^{-\frac{1}{n-t}}. \tag{5.2.1}$$

Moreover, observe that

$$2^{i-1} \in [c_1\Pi_t(V), \frac{c_0}{2}\Pi_{t+1}(V)] \iff \log_2 c_1\Pi_t(V) + 1 \leq i \leq \log_2 \frac{c_0}{2}\Pi_{t+1}(V) + 1.$$

We can set $c_1 = \frac{1}{\Pi_{i_V(M)}(V)}$ and since $1 \leq t \leq i_V(M) - 1$ we find that in this case

$$1 \leq i \leq \log_2 \frac{c_0}{2}\Pi_{i_V(M)}(V) + 1.$$

Therefore using (5.2.1) we get the bound

$$\begin{aligned}
\sum_{i=1}^{1+\log_2 \frac{c_0}{2}\Pi_{i_V(M)}(V)} \deg(h_i) &\lesssim_n \sum_{t=0}^{i_V(M)-1} \sum_{i=\log_2 c_1\Pi_t(V)}^{1+\log_2 \frac{c_0}{2}\Pi_{i_V(M)}(V)} \Delta_t(V)^{-\frac{1}{n-t}} 2^{\frac{i-1}{n-t}} \\
&\lesssim_n \sum_{t=0}^{i_V(M)-1} \Delta_t(V)^{-\frac{1}{n-t}} \left(\delta_{t+1}(V)^{n-(t+1)}\Delta_{t+1}(V)\right)^{\frac{1}{n-t}} \\
&\lesssim_n \sum_{t=0}^{i_V(M)-1} \delta_{t+1}(V) \lesssim_n M.
\end{aligned}$$

If we can not find an admissible integer $t < i_V(M)$ such that $2^{i-1} \in [c_1\Pi_t(V), \frac{c_0}{2}\Pi_{t+1}(V)]$, we can estimate the degree as follows. Recall that $i_V(M)$ is the smallest admissible integer such that

$$r = M^{n-i_V(M)}\Delta_{i_V(M)}(V) \in \mathcal{R}_{i_V(M)}(V).$$

This means that

$$c_1\Pi_{i_V(M)}(V) \leq r \leq \frac{c_0}{2}\delta_{i_V(M)+1}(V)^{n-i_V(M)}\Delta_{i_V(M)}(V).$$

Since $i \leq \log_2 r$ we have that $2^{i-1} \leq \frac{r}{2}$. Since $c_1 = \frac{1}{\Pi_{i_V(M)}(V)}$ we can guarantee that $2^{i-1} \in \mathcal{R}_{i_V(M)}(V)$. By Theorem 5.1.1 we can bound the degree of the remaining h_i using the bound

$$\deg(h_i) \lesssim_n 2^{\frac{i-1}{n-i_V(M)}}\Delta_{i_V(M)}(V)^{-\frac{1}{n-i_V(M)}}.$$

Using the previous bound we obtain

$$\begin{aligned}\sum_{1+\log_2 \frac{c_0}{2}\Pi_{i_V(M)}(V)}^{\log_2 r} \deg(h_i) &\lesssim_n \Delta_{i_V(M)}(V)^{-\frac{1}{n-i_V(M)}} \sum_{1+\log_2 \frac{c_0}{2}\Pi_{i_V(M)}(V)}^{\log_2 r} 2^{\frac{i-1}{n-i_V(M)}} \\ &\lesssim_n \Delta_{i_V(M)}(V)^{-\frac{1}{n-i_V(M)}} r^{\frac{1}{n-i_V(M)}} \lesssim_n M.\end{aligned}$$

This concludes the proof. □

With the above theorem, the proof of Theorem 1.0.14 is almost immediate.

Proof of Theorem 1.0.14. Notice that $n-d$ is an admissible integer since, by convention $\delta_i(V) = \infty$ for all $i > n-d$. Since $M \geq \delta(V)$ we can take $i_V(M) = n-d$. In this case $\Delta_{i_V(M)} = \deg(V)$ and the result follows from Theorem 5.2.2. □

Chapter 6

Bounds for the number of connected components intersected by a variety

In chapter 2 we saw how to prove the Szemerédi-Trotter theorem (Theorem 2.2.9) using the polynomial partitioning theorem (Theorem 2.2.5) together with the following fact: a line can only touch at most $D+1$ of the cells of $\mathbb{R}^2 \setminus Z(P)$ where $P \in \mathbb{R}[x,y]_{\leq D}$.
In this chapter we want to prove Theorem 1.0.15 which is the analog of the above fact for real points of an irreducible algebraic variety. To this end, we will introduce the concept of *envelope* of an irreducible algebraic variety and prove useful estimates regarding the degree of its higher dimensional irreducible components. We will make further use of part of the work done in this chapter at the last chapter, when discussing how to bound the number of connected components of an algebraic set.

6.1 Envelopes

Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d and let $f_1, \ldots, f_{n-d}$ be polynomials such that $V \subseteq Z(f_1, \ldots, f_{n-d})$. Using the results of Barone and Basu [BB15] we can obtain bounds for the number of connected components of $Z(f_1, \ldots, f_{n-d})(\mathbb{R})$ if the points of the components have local real dimension $\leq n-i$ in $Z(f_1, \ldots, f_i)$ for every i. The *envelope* of V over the polynomials $f_1, \ldots, f_{n-d}$ allows us to control the irreducible components where we can not use the results of [BB15].
In order to get the best result possible we will need to choose a suitable set of polynomials.

Definition 6.1.1 (Admissible tuple)**.** Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d and $1 \leq K_1 \leq \cdots \leq K_{n-d}$ real numbers. We say an ordered tuple of polynomials $Q = \{Q_1, \ldots, Q_{n-d}\}$ is $(K_1, \ldots, K_{n-d})$-*admissible for* V if, for every $1 \leq i \leq n-d$, we have that $\deg(Q_i) \leq K_i\delta_i(V)$, the maximal dimension of an irreducible component of $Z(Q_1, \ldots, Q_i)$ containing V is equal to $n-i$ and $Z(Q_1, \ldots, Q_i)$ contains an $(n-i)$-minimal variety of V. If Q is $(K_1, \ldots, K_{n-d})$-admissible for V and $K \geq K_{n-d}$ then we may simply say Q is K-admissible for V.

Definition 6.1.2 (Envelopes)**.** Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d and $Q = \{Q_1, \ldots, Q_{n-d}\}$ a K-admissible tuple of polynomials for V, for some $K \geq 1$. For each $1 \leq j \leq n-d$, we define the *j-th envelope of V over Q* to be the union of all irreducible components of $Z(Q_1, \ldots, Q_j)$ of dimension strictly greater than $n-j$ and write $\mathcal{E}_V^{(j)}(Q)$ for this algebraic set. We also write

$$\mathcal{E}_V(Q) = \bigcup_{j=1}^{n-d} \mathcal{E}_V^{(j)}(Q),$$

and call this algebraic set the envelope of V over Q. Finally, for every $1 \leq j \leq n-d$, we write $\mathcal{S}_V^{(j)}(Q)$ for the algebraic set given by the union of the irreducible components of $Z(Q_1, \ldots, Q_j)$ of dimension $n-j$.

We will need a couple of observations.

Lemma 6.1.3. *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d and $Q = \{Q_1, \ldots, Q_{n-d}\}$ a $(K_1, \ldots, K_{n-d})$-admissible tuple of polynomials for V. Let W be an irreducible component of $Z(Q_1, \ldots, Q_i)$ of dimension $n-i$, for some $1 \leq i \leq n-d$. Then, if $\deg(W) \geq c\delta_1(V) \cdots \delta_i(V)$, it must be $\delta_j(W) \sim_{K_i,c,n} \delta_j(V)$ for every $1 \leq j \leq i$.*

Lemma 6.1.4. *If W is an irreducible component of $\mathcal{E}_V(Q)$ of dimension i, then it is also an irreducible component of $\mathcal{S}_V^{(n-i)}(Q)$.*

Proof. Suppose that there exists an irreducible component $W' \subseteq Z(Q_1, \ldots, Q_{n-i})$ which properly contains W. Since W has dimension i the variety W' must have dimension strictly larger than i, thus by definition of $(n-i)$-th envelope

$$W' \in \mathcal{E}_V^{n-i}(Q) \subseteq \mathcal{E}_V(Q).$$

This is a contradiction since W is an irreducible component of $\mathcal{E}_V(Q)$. □

The next proposition tells us that given an algebraic variety V we can always find an admissible tuple $\{Q_1, \ldots, Q_{n-d}\}$ such that the higher dimensional components of $Z(Q_1, \ldots, Q_i)$ have their degrees controlled by the partial degrees of V for every i. Lemma 6.1.6 will tell us that we can also find polynomials vanishing on these components.

Proposition 6.1.5. *Let $V \subseteq \mathbb{C}^n$ be a d-dimensional variety and let $\epsilon_1 \geq \cdots \geq \epsilon_{n-d-1} > 0$ be given. Then, there exist constants $C_1, \ldots, C_{n-d}$ with $C_i = O_{n,\epsilon_{i-1}}(1)$ and a $(C_1, \ldots, C_{n-d})$-admissible tuple of polynomials $Q = \{Q_1, \ldots, Q_{n-d}\}$ for V, such that, for every $1 \leq i \leq n-d-1$, the union of all irreducible components of $\mathcal{E}_V(Q)$ of dimension $n-i$ has degree less than $\epsilon_i\delta_1(V) \cdots \delta_i(V)$.*

Proof. We are going to recursively construct the polynomials $Q_1, \ldots, Q_{n-d}$. Let us see first how we can guarantee that the polynomials $Q_1, \ldots, Q_{n-d}$ for an admissible tuple.

The definition of admissible tuple specifies that, for every $1 \leq i \leq n-d$ these polynomials have to satisfy $\deg(Q_i) \leq C_i \delta_i(V)$, the maximal dimension of an irreducible component of $Z(Q_1, \ldots, Q_i)$ containing V is equal to $n-i$ and $Z(Q_1, \ldots, Q_i)$ contains a $(n-i)$-minimal variety of V. By Lemma 3.2.6 we know that there exist polynomials $P_1, \ldots, P_{n-d}$ such that for every $1 \leq i \leq n-d$ there is an irreducible component of $Z(P_1, \ldots, P_i)$ of dimension $n-i$ containing V that is an $(n-i)$-minimal variety of V. Therefore, if for every $1 \leq i \leq n-d$ we construct $Z(Q_1, \ldots, Q_i)$ in such a way that $\deg(Q_i) \leq C_i \delta_i(V)$, the maximal dimension of an irreducible component of $Z(Q_1, \ldots, Q_i)$ containing V is equal to $n-i$ and $Z(Q_1, \ldots, Q_i)$ contains every $(n-i)$-dimensional component of $Z(P_1, \ldots, P_i)$ containing V then $Z(Q_1, \ldots, Q_{n-d})$ will be an admissible tuple for V.

Finally, using Lemma 6.1.4 we can prove that the union of all irreducible components of $\mathcal{E}_V(Q)$ of dimension $n-i$ has degree less than $\epsilon_i \delta_1(V) \cdots \delta_i(V)$ by showing that Q_i does not vanish identically on any union of components of $\mathcal{S}_V^{i-1}(Q)$ having degree at least $\epsilon_{i-1} \delta_1(V) \ldots \delta_{i-1}(V)$.

Notice that we can start by taking $Q_1 = P_1$ since the required conditions specified above are guaranteed to hold for P_1 by Lemma 3.2.6. Suppose that we have constructed the required polynomials up to $i-1$.

By Lemma 3.2.6, the maximal dimension of an irreducible component of $Z(P_1, \ldots, P_i)$ containing V is $n-i$. Therefore all of the polynomials $P_1, \ldots, P_i$ vanish on V without vanishing at some of the $(n-i+1)$-dimensional irreducible components of $Z(Q_1, \ldots, Q_{i-1})$. This means that we can apply Lemma 3.2.5 to find a polynomial vanishing on V without vanishing on any of the $(n-i+1)$-dimensional irreducible components of $Z(Q_1, \ldots, Q_{i-1})$. Therefore the highest dimension of an irreducible component of $Z(Q_1, \ldots, Q_{i-1}, f)$ containing V is $n-i$. Furthermore, every irreducible component of $Z(P_1, \ldots, P_i)$ of dimension $n-i$ containing V is contained in $Z(Q_1, \ldots, Q_{i-1}, f)$.

Let A be the algebraic set consisting on the union of all irreducible components of $W \subseteq Z(Q_1, \ldots, Q_{i-1}, f)$ such that $\dim(W) = n-i$ and $V \subseteq W$.

Since $Z(Q_1, \ldots, Q_{i-1}, f) \cap A = A$ and $\deg(f) \leq \delta_i(V)$, by Lemma 3.1.5 we find

$$\deg(A) \leq (\prod_{j=1}^{i-1} C_j) \delta_1(V) \cdots \delta_i(V).$$

Moreover, by Corollary 3.2.13 we know that

$$\delta_1(V) \cdots \delta_i(V) \lesssim_n \deg(A).$$

Let $W_1, \ldots, W_r$ be the irreducible components of $\mathcal{S}_V^{i-1}(Q)$. Let us consider an arbitrary subset $J \subseteq \{1, \ldots, r\}$ and let

$$W_J := \bigcup_{j \in J} W_j.$$

Suppose that $\deg(W_J) \geq \epsilon_{i-1}\delta_1(V)\ldots\delta_{i-1}(V)$. Observe that since each W_j is an irreducible component of $Z(Q_1,\ldots,Q_{i-1})$ we know that $\delta(W_J) \lesssim_{n,\epsilon_{i-2}} \delta_{i-1}(V)$. Moreover, we get

$$\deg(W_J) \leq \prod_{j=1}^{i-1}\deg(Q_j) \lesssim_{n,\epsilon_{i-2}} \prod_{j=1}^{i-1}\delta_j(V).$$

We conclude that

$$\deg(A) \gtrsim_{n,\epsilon_{i-2}} \deg(W_J)\delta_i(V) \gtrsim_{n,\epsilon_{i-2}} \deg(W_J)\delta(W_J).$$

Using Theorem 4.0.3 we find a polynomial f_J of degree

$$\lesssim_{n,\epsilon_{i-2}} \frac{\deg(A)}{\deg(W_J)} \lesssim_{n,\epsilon_{i-1}} \delta_i(V),$$

vanishing on A without vanishing identically on W_J. By Lemma 3.2.5 we know that there exists a linear combination Q_i of polynomials f_J over all J with $\deg(W_J) \geq \epsilon_{i-1}\delta_1(V)\cdots\delta_{i-1}(V)$ such that Q_i vanishes on A without vanishing identically on any W_J. □

Lemma 6.1.6. *Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety of dimension d. Then there exists an $O_n(1)$-admissible tuple $Q = \{Q_1,\ldots,Q_{n-d}\}$ of polynomials for V and polynomials $F_1,\ldots,F_{n-d-1}$, such that F_k vanishes on all the irreducible components of $\mathcal{E}_V(Q)$ of dimension $n-k$ without vanishing identically on V.*

Proof. Let $\epsilon_1 > \cdots > \epsilon_{n-d-1} > 0$ be real numbers such that each ϵ_i is sufficiently small with respect to n and ϵ_{i-1}. By Proposition 6.1.5 we know that there exists a $(C_1,\ldots,C_{n-d})$-admissible tuple of polynomials $Q = \{Q_1,\ldots,Q_{n-d}\}$ such that $C_k = O_{n,\epsilon_{k-1}}(1)$ and the algebraic set

$$A_k := \bigcup_{W\subseteq\mathcal{E}_V(Q)} W$$

satisfies

$$\deg(A_k) \leq \epsilon_i\delta_1(V)\cdots\delta_{n-k}(V) \tag{6.1.1}$$

for all $1 \leq k \leq n-d$. Moreover, by definition of admissible tuple we know that $Z(Q_1,\ldots,Q_{k-1})$ contains a $(n-k+1)$-minimal variety V_{n-k+1} of V for all $1 \leq k \leq n-d$. Since A_k is a finite set of $(n-k)$-dimensional irreducible algebraic varieties in $\mathbb{C}^n$ and $\dim(V_{n-k+1}) = n-k+1$ we can use Theorem 4.0.3 to find a polynomial F_k of degree

$$\lesssim_n \left(\frac{\deg(A_k)}{\Delta_s(V_{n-k+1})}\right)^{\frac{1}{k-s}} \tag{6.1.2}$$

such that F_k vanishes on A_k without vanishing identically on V_{n-k+1}. By Corollary 3.2.13 we know that

$$\deg(V_{n-k+1}) \gtrsim_n \delta_1(V)\cdots\delta_{k-1}(V).$$

and by Lemma 6.1.3 we find

$$\delta_j(V_{n-k+1}) \sim_{n,\epsilon_{k-1}} \delta_j(V_{n-k+1}) \text{ for all } 1 \leq j \leq k-1.$$

thus,

$$\Delta_s(V_{n-k+1}) \sim_{n,\epsilon_{k-1}} \delta_1(V)\cdots\delta_s(V). \tag{6.1.3}$$

From the bounds (6.1.3) and (6.1.2) we get that

$$\deg(F_k) \lesssim_{n,\epsilon_{k-1}} \left(\frac{\epsilon_k\delta_1(V)\cdots\delta_k(V)}{\delta_1(V)\cdots\delta_s(V)}\right)^{\frac{1}{k-s}} \leq (\epsilon_k\delta_k(V)^{k-s})^{\frac{1}{k-s}} = \epsilon_k^{\frac{1}{k-s}}\delta_k(V).$$

Notice that $\epsilon_k^{\frac{1}{k-s}}\delta_k(V) < \delta_k(V)$ if ϵ_k is sufficiently small with respect to n and ϵ_{k-1}. Therefore $\deg(F_k)$ is strictly smaller than $\delta_k(V)$. Observe that F_k cannot vanish on V since otherwise $F_k \in I(V) \setminus I(V_{n-k+1})$ and by definition of $(n-k+1)$-minimal variety F_k would have degree at least $\delta_k(V)$. □

We will need the following two results in chapter 8.

Lemma 6.1.7. *Let $W_1, \cdots, W_r$ be the irreducible components of $\mathcal{S}_V^{(i)}(Q)$ for some $1 \leq i \leq n-d$. Then*

$$\sum_{i=1}^{r} \deg(W_i) \lesssim_{K,n} \delta_1(V)\cdots\delta_i(V).$$

Proof. By definition we know that every irreducible component of $\mathcal{S}_V^{(i)}(Q)$ is an irreducible component of $Z_i(Q)$. Using Lemma 3.1.5 we get the bound

$$\sum_{i=1}^{r} \deg(W_i) \leq \sum_{W\subseteq Z_i(Q)} \deg(W) \leq \prod_{j=1}^{i} \deg(Q_j) \leq \prod_{j=1}^{i} K\delta_j(V),$$

where the second sum is taken over all irreducible components of $Z(Q_1, \ldots, Q_i)$. □

Corollary 6.1.8. *Let $W_1, \ldots, W_r$ be the irreducible components of $\mathcal{E}_V(Q)$ of dimension $n-k$. Then*

$$\sum_{i=1}^{r} \deg(W_i) \lesssim_{K,n} \delta_1(V)\cdots\delta_k(V).$$

Proof. By Lemma 6.1.4 we know that each irreducible component of $\mathcal{E}_V(Q)$ of dimension $n-k$ is also an irreducible component of the algebraic set $\mathcal{S}_V^{(k)}(Q)$. Using Lemma 6.1.7 we obtain the desired bound. □

6.2 Previously known bounds

In this small section we want to introduce in more detail the results of Barone and Basu [BB15]. For this we will need to following notation.

Notation 6.2.1. Given an algebraic set $X \subseteq \mathbb{C}^n$ and a point $x \in X(\mathbb{R})$, we write $\dim_x^{\mathbb{R}} X$ for the *local real dimension of* $X(\mathbb{R})$ *at* x.

Notation 6.2.2. If $Q = \{Q_1, \ldots, Q_m\}$, $Q_1, \ldots, Q_m \in \mathbb{C}[x_1, \ldots, x_n]$, is a tuple of polynomials with $m \leq n$, then for every $1 \leq j \leq m$ we will write

$$Z_j(Q) := Z(Q_1, \ldots, Q_j).$$

Notation 6.2.3. If $x \in Z_j(Q)(\mathbb{R})$, we define

$$\dim^{\mathbb{R}}_{Q,(j)}(x) := \Big(\dim_x^{\mathbb{R}} Z_1(Q), \ldots, \dim_x^{\mathbb{R}} Z_j(Q) \Big).$$

Notation 6.2.4. Given a j-tuple τ of non-negative integers with $1 \leq j \leq m$ and a set of polynomials $Q = \{Q_1, \ldots, Q_m\}$ we define

$$Z_\tau(Q) := \overline{\{x \in Z_j(Q)(\mathbb{R}) \ : \ \dim^{\mathbb{R}}_{Q,(j)} = \tau\}}$$

where the closure is taken over $\mathbb{R}^n$.

Similar notation was introduced in [BB15, Section 3], where we can find examples motivating it.

Proposition 6.2.5 (Barone-Basu [BB15])**.** *Let* $Q = \{Q_1, \ldots, Q_m\}$*,* $Q_1, \ldots, Q_m \in \mathbb{Q}[x_1, \ldots, x_n]$*,* $1 \leq m \leq n$*, be a set of polynomials and* $d_1 \leq \cdots \leq d_m$ *integers with* $\deg(Q_i) \leq d_i$ *for every* i*. Let* $1 \leq j \leq m$ *and let* $\tau = (\tau_1, \ldots, \tau_j)$*,* $\tau_1 \geq \cdots \geq \tau_j$*, be a* j*-tuple of non-negative integers satisfying* $\tau_i \leq n - i$ *for every* $1 \leq i \leq j$*. Then, the number of connected components of* $Z_j(Q)(\mathbb{R})$ *intersecting* $Z_\tau(Q)$ *is at most*

$$\lesssim_n (\prod_{i=1}^{j} d_i) d_j^{n-j}.$$

We will be interested in deducing what bounds we get if Q is an admissible tuple. For this we will make use of the following observation.

Lemma 6.2.6. *Let* $V \subseteq \mathbb{C}^n$ *be an irreducible variety of dimension* d *and* Q *a* K*-admissible tuple of polynomials for* V*. Let* $x \in Z_j(Q)(\mathbb{R}) \setminus \mathcal{E}_V^{(j)}(Q)$*. Then*

$$\dim^{\mathbb{R}}_{Q,(j)}(x) \leq (n-1, n-2, \ldots, n-j).$$

Corollary 6.2.7. *Let* $V \subseteq \mathbb{C}^n$ *be an irreducible variety of dimension* d *and* Q *a* K*-admissible tuple of polynomials for* V*. Let* $1 \leq j \leq n-d$ *and let* $\tau = (\tau_1, \ldots, \tau_j)$*,* $\tau_1 \geq \cdots \geq \tau_j$*, be a* j*-tuple of non-negative integers satisfying* $\tau_i \leq n - i$ *for every* $1 \leq i \leq j$*. Then, the number of connected components of* $Z_j(Q)(\mathbb{R})$ *intersecting* $Z_\tau(Q)$ *is at most* $\lesssim_{n,K} \delta(V)^d \deg(V)$*.*

Proof. By definition of admissible tuple we have that $\deg(Q_i) \leq K\delta_i(V)$ for every $1 \leq i \leq n-d$. Therefore, by Proposition 6.2.5 we know that the number of connected components of $Z_j(Q)(\mathbb{R})$ intersecting $Z_\tau(Q)$ is at most

$$\lesssim_n (\prod_{i=1}^{j} K\delta_i(V))(K\delta_j(V))^{n-j} \leq (\prod_{i=1}^{n-d} K\delta_i(V))(K\delta_{n-d}(V))^d \lesssim_n K^n \deg(V)\delta(V)^d,$$

as we wanted to see. □

6.3 Proof of the main theorem

We can now give a proof of the main theorem of the chapter.

Proof of Theorem 1.0.15. We proceed by induction on the dimension of V. Let $\mathcal{S} \subset \mathbb{R}^n$ be a finite set of points and let $P \in \mathbb{R}[x_1, \dots, x_n]$. Then it is clear that $\mathcal{S}$ intersects at most $|\mathcal{S}|$ connected components of $\mathbb{R}^n \setminus Z(P)$ so we are done for the case $d = 0$.
Let $d > 0$ and suppose that the result is true for all varieties of smaller dimension. If $\deg(P) \leq \delta_1(V)$ it is already know (see for example [Mil64, Tho65]) that the number of connected components of $\mathbb{R}^n \setminus Z(P)$ is bounded by

$$\lesssim_n \deg(P)^n \lesssim_n \delta_1(V) \cdots \delta_{n-d}(V) \deg(P)^d \lesssim_n \deg(V) \deg(P)^d.$$

Let us then suppose that $\deg(P) \geq \delta_j(V)$ where $\delta_j(V)$ is the smallest partial degree achieving this bound. Let Q be an $O_n(1)$-admissible tuple for V of the form provided by Lemma 6.1.6. We start by studying the points inside of $(V \cap \mathcal{E}_V^{(j)}(Q))(\mathbb{R})$.
Let $F_1, \dots, F_{n-d-1}$ be the polynomials associated to Q by Lemma 6.1.6. We know that F_i is a polynomial of degree $< \delta_i(V)$ which vanishes on all the irreducible components of $\mathcal{E}_V(Q)$ of dimension $n - i$ without vanishing identically on V. This means that the polynomial

$$F = \prod_{i=1}^{j-1} F_i$$

has degree

$$\lesssim_n \delta_{j-1}(V) \lesssim_n \deg(P)$$

and vanishes on all the irreducible components of $\mathcal{E}_V(Q)$ of dimension $> n - j$. In particular, F vanishes on $\mathcal{E}_V^j(Q)$ without vanishing identically on V.
Therefore, it is enough to estimate the number of connected components of $\mathbb{R}^n \setminus Z(P)$ intersected by $V \cap Z(F)$. Let $W_1, \dots, W_r$ be the irreducible components of this intersection. Since $F \notin I(V)$ we know that $\dim(W_i) = d - 1$ for all i. By induction we obtain that the number of connected components of $\mathbb{R}^n \setminus Z(P)$ instersected by W_i is bounded by

$$\lesssim_n \deg(W_i) \deg(P)^{\dim(W_i)}.$$

Taking the sum over all the components and using Lemma 3.1.5 we obtain the following bound for the number of components of $\mathbb{R}^n \setminus Z(P)$ intersected by $V \cap Z(F)$:

$$\lesssim_n \sum_{i=1}^{r} \deg(W_i)\deg(P)^{\dim(W_i)} \lesssim_n \deg(V)\deg(F)\deg(P)^{d-1} \lesssim_n \deg(V)\deg(P)^d.$$

We now need to bound the number of connected components of $\mathbb{R}^n \setminus Z(P)$ intersected by $V(\mathbb{R}) \setminus Z(F)$.
Consider the polynomial $f := PF\overline{F}$. Since $V \subseteq Z_j(Q)$ it suffices to bound the number of connected components of $\mathbb{R}^n \setminus Z(f) = (\mathbb{R}^n \setminus Z(P)) \setminus Z(F)$ intersected by $Z_j(Q)$. Without loss of generality we can also bound the number of these components where $f > 0$. We are going to distinguish two cases.
Let $\mathcal{C}$ be the set of all connected components C of $\mathbb{R}^n \setminus Z(f)$ such that there exists $x \in Z_j(Q) \cap C$ with $f(x) > 0$ that can be joined to $Z(f)$ through a path π_C in $Z_j(Q)(\mathbb{R})$. Clearly we can suppose that only the end point of this path lies inside of $Z(f)$. Let π_C^0 denote the path π_C without the end point.

Claim 6.3.1. *Let $C \in \mathcal{C}$. If $x \in \pi_C^0$ then $x \in (Z_1 \cup \cdots \cup Z_m) \cap Z_\tau(Q)$ for some j-tuple $\tau \leq (n-1, n-2, \ldots, n-j)$ depending on x.*

Proof. Since $x \in \pi_C^0$ and $\mathcal{E}_V^{(j)}(Q) \subseteq Z(F)$ then $x \in Z_j(Q)(\mathbb{R}) \setminus \mathcal{E}_V^{(j)}(Q)$. Moreover, all the irreducible components of $Z_j(Q)$ are of dimension $\geq n-j$. Therefore by definition of $\mathcal{E}_V^{(j)}(Q)$ and the fact that Z_i is not contained in $Z(f)$ for any i we conclude that

$$Z_j(Q)(\mathbb{R}) \setminus \mathcal{E}_V^{(j)}(Q) \subseteq Z_1 \cup \cdots \cup Z_m.$$

That $x \in Z_\tau(Q)$ follows immediately from Lemma 6.2.6. □

Notice that for each $C \in \mathcal{C}$ there exists a path from an element of $Z_j(Q)(\mathbb{R}) \cap C$ to $Z(f)$, hence we can find some $\epsilon_C > 0$ such that

$$(0, \epsilon_C) \subset f((Z_1 \cup \cdots \cup Z_m) \cap C).$$

Since there are finitely many connected components $\mathbb{R}^n \setminus Z(f)$ there exists some $\epsilon' > 0$ such that

$$(0, \epsilon') \subset f((Z_1 \cup \cdots \cup Z_m) \cap C)$$

for every $C \in \mathcal{C}$. We now need two observations. First, since there are finitely many irreducible components $Z_1, \ldots, Z_m$ we can always find some $0 < \epsilon < \epsilon'$ such that the polynomial $f - \epsilon$ does not vanish on any Z_i. The other observations is that for every $C \in \mathcal{C}$ there exists a point $x \in (Z_1 \cup \cdots \cup Z_m) \cap C$ such that $f(x) = \epsilon$ with $0 < \epsilon < \epsilon'$. Taking into account both facts we conclude that there exists some $0 < \epsilon < \epsilon'$ such that the algebraic set

$$X = (Z_1 \cup \cdots \cup Z_m) \cap Z(f - \epsilon)$$

intersects every element of $\mathcal{C}$ and $f - \epsilon$ does not vanish on any Z_i.
By the claim and the fact that $f - \epsilon$ does not vanish on any Z_i we know that every $x \in$

X belongs to $Z_\sigma(\{Q_1, \ldots, Q_j, f-\epsilon\})$ for some $(j+1)$-tuple $\sigma \leq (n-1, \ldots, n-j-1)$ depending on x. We conclude that $|\mathcal{C}|$ can be bounded by the number of connected components of $Z(Q_1, \ldots, Q_j, f-\epsilon)$ intersecting some $Z_\sigma(\{Q_1, \ldots, Q_j, f-\epsilon\})$ of the above form.
Since Q is an $O_n(1)$-admissible tuple we know that $Q_i \leq \delta_i(V)$ for every $1 \leq i \leq j$. We also know that $\deg(f) \lesssim_n \deg(P)$. Therefore by Proposition 6.2.5 we get the bound

$$|\mathcal{C}| \lesssim_n \delta_1(V) \cdots \delta_j(V) \deg(f) \deg(f)^{n-j-1} \lesssim_n \deg(V) \deg(P)^d.$$

We finish the proof by bounding the number of connected components C of $\mathbb{R}^n \setminus Z(f)$ intersected by $Z_j(Q)$ but such that no element of $Z_j(Q) \cap C$ can be joined to $Z(f)$ through a path inside of $Z_j(Q)(\mathbb{R})$. We will call the set of these components $\mathcal{C}'$.
If $C \in \mathcal{C}'$ then there is some connected component Z_C of $Z_j(Q)(\mathbb{R})$ properly contained in C. Since $\mathcal{E}_V^{(j)}(Q) \subseteq Z(F)$ we know that Z_C contains an element of $Z_j(Q)(\mathbb{R}) \setminus \mathcal{E}_V^{(j)}(Q)$. Therefore Z_C intersects $Z_\tau(Q)$ for some $\tau \leq (n-1, \ldots, n-j)$. By Corollary 6.2.7, the number of such components is at most

$$\lesssim_n \delta_1(V) \cdots \delta_j(V) \delta_j(V)^{n-j} \lesssim_n \deg(V) \delta(V)^d$$

and we are done. □

Chapter 7

An incidence estimate for hypersurfaces over general varieties

We devote this section to the proof of Theorem 1.0.18. We start by introducing the concept of a (k, b)-*free set* and by proving a weak bound for the number of incidences between a set of points $\mathcal{S} \subset \mathbb{R}^n$ and a set of varieties $\mathcal{T}$.
As we did in chapter 2 for the proof of Theorem 2.2.9, we will use this estimate to bound the incidences within the connected components created using Theorem 5.2.2.

Definition 7.0.1. Let $\mathcal{S}$ be a finite set of points in $\mathbb{R}^n$ and $\mathcal{T}$ a finite set of varieties in $\mathbb{R}^n$. We say $\mathcal{S}$ is (k, b)-*free with respect to* $\mathcal{T}$ if, for every choice of k distinct elements $s_1, \ldots, s_k$ from $\mathcal{S}$ and b distinct elements $t_1, \ldots, t_b$ from $\mathcal{T}$, we have $\mathcal{S}_i \notin t_j$ for some $1 \leq i \leq k$, $1 \leq j \leq b$.

Lemma 7.0.2. *Let $\mathcal{S}$ be a finite set of points in $\mathbb{R}^n$ and $\mathcal{T}$ a finite set of varieties in $\mathbb{R}^n$. Let $k, b \geq 1$ be integers such that $\mathcal{S}$ is (k, b)-free with respect to $\mathcal{T}$. Then*

$$I(\mathcal{S}, \mathcal{T}) \leq b^{1/k} |\mathcal{S}| |\mathcal{T}|^{1-1/k} + (k-1)|\mathcal{T}|$$

Proof. We proceed by induction on k. If $\mathcal{S}$ is $(1, b)$-free with respect to $\mathcal{T}$ then for every $s \in S$ and every choice of b distinct elements $t_1, \ldots, t_b$ of $\mathcal{T}$ there exists $j \in \{1, \ldots, b\}$ such that $s \notin t_j$. Therefore there are at most $(b-1)$ incidences for each point of $\mathcal{S}$ and each choice of elements of $\mathcal{T}$. This gives us the bound

$$I(\mathcal{S}, \mathcal{T}) \leq (b-1)|\mathcal{S}| \leq b|\mathcal{S}|$$

proving the base case. Let $k > 1$. Suppose that $\mathcal{S}$ is (k, b)-free with respect to $\mathcal{T}$ and that the result has been proven for every smaller k. For every $s \in \mathcal{S}$ let us define the set $T_s := \{t \in T \mid s \in t\}$ and for each $t \in \mathcal{T}$ define $S_t := \{s \in S \mid s \in t\}$. Notice that

$$I(\mathcal{S}, \mathcal{T}) = \sum_{s \in S} |T_s| = \sum_{t \in T} |S_t|. \tag{7.0.1}$$

We will also need the following observation. Take k distinct points $s_1, \ldots, s_{k-1}, s$ of $\mathcal{S}$ and b distinct elements $t_1, \cdots, t_b$ of T_s. Since $\mathcal{S}$ is (k,b)-free with respect to $\mathcal{T}$ we have that $s_i \notin t_j$ for some $1 \leq i \leq k-1$ and $1 \leq j \leq b$. This means that for each $s \in \mathcal{S}$ the set $\mathcal{S} \setminus \{s\}$ is $(k-1,b)$-free with respect to T_s.
We now do double counting on $\sum_{s\in\mathcal{S}} I(\mathcal{S}, T_s)$. First of all, observe that $I(\mathcal{S}, T_s) = |T_s| + I(\mathcal{S}\setminus\{s\}, T_s)$. Using this fact, the induction hypothesis and (7.0.1) we obtain the following bound:

$$\begin{aligned}
\sum_{s\in\mathcal{S}} I(\mathcal{S}, T_s) &\leq \sum_{s\in\mathcal{S}} I(\mathcal{S}\setminus\{s\}, T_s) + \sum_{s\in\mathcal{S}} |T_s| \\
&\leq \sum_{s\in\mathcal{S}} b^{\frac{1}{k-1}}(|\mathcal{S}|-1)|T_s|^{1-\frac{1}{k-1}} + \sum_{s\in\mathcal{S}}(k-2)|T_s| + \sum_{s\in\mathcal{S}} |T_s| \\
&\leq \sum_{s\in\mathcal{S}} b^{\frac{1}{k-1}}|\mathcal{S}||T_s|^{1-\frac{1}{k-1}} + (k-1)\sum_{s\in\mathcal{S}} |T_s| \\
&\leq b^{\frac{1}{k-1}}|\mathcal{S}|^{1+\frac{1}{k-1}} I(\mathcal{S},\mathcal{T})^{1-\frac{1}{k-1}} + (k-1) I(\mathcal{S},\mathcal{T}).
\end{aligned}$$

We can also obtain a lower bound as follows.

$$\sum_{s\in\mathcal{S}} I(\mathcal{S}, T_s) \geq \sum_{t\in T} |S_t|^2 \geq \frac{1}{|\mathcal{T}|}\left(\sum_{t\in T}|S_t|\right)^2 \geq \frac{I(\mathcal{S},\mathcal{T})^2}{|\mathcal{T}|}.$$

Comparing both bounds we get the following inequality:

$$\frac{I(\mathcal{S},\mathcal{T})^2}{|\mathcal{T}|} \leq b^{\frac{1}{k-1}}|\mathcal{S}|^{1+\frac{1}{k-1}} I(\mathcal{S},\mathcal{T})^{1-\frac{1}{k-1}} + (k-1) I(\mathcal{S},\mathcal{T}).$$

□

Using the tools introduced in chapter 5 and chapter 6 together with the previous estimate we can prove Theorem 1.0.18.

Notation 7.0.3. We will use the following quantities.

$$\alpha_k(d) = \frac{k(d-1)}{dk-1}, \quad \beta_k(d) = \frac{d(k-1)}{dk-1} \quad \text{and} \quad \tau_d(b,k) = b^{1-\beta_k(d)} k^{1-\alpha_k(d)}.$$

We set $\alpha_1(1) = 0$ and $\beta_1(1) = 1$.

Proof of Theorem 1.0.18. The proof is by induction on the dimension of V. Let $d = 1$ and $|\mathcal{S}| < k$ then $I(\mathcal{S},\mathcal{T}) \leq (k-1)\deg(\mathcal{T})$. On the other hand, suppose that $|\mathcal{S}| \geq k$. Since $\mathcal{S}$ is (k,b)-free with respect to $\mathcal{T}$ we can find at most $(b-1)$ different hypersurfaces in $\mathcal{T}$ containing V. Let us call the set of these hypersurfaces $\mathcal{T}_V$. It is clear that

$$I(\mathcal{S},\mathcal{T}_V) \leq (b-1)|\mathcal{S}|.$$

For the remaining hypersurfaces we can use Lemma 3.1.5. Since the dimension of $t \cap V$ is zero we have $\deg(t\cap V) = |t \cap V|$. Let $W_1, \ldots, W_r$ be the irreducible components of $t \cap V$. Since all the dimensions of these irreducible components are equal (in particular, we have that $\dim(W_i) = 0$ for all $1 \le i \le r$) then $\deg(t \cap V) = \sum_{i=1}^{r} W_i$. Finally, let h be the defining polynomial of the hypersurface $t \in \mathcal{T} \setminus \mathcal{T}_V$ then by Lemma 3.1.5 we find the bound

$$|t \cap V| = \sum_{i=1}^{r} \deg(W_i) \le \deg(V)\deg(h) = \deg(V)\deg(t).$$

Therefore

$$I(\mathcal{S}, \mathcal{T} \setminus \mathcal{T}_V) = \sum_{t \in \mathcal{T} \setminus \mathcal{T}_V} |t \cap V| \le \sum_{t \in \mathcal{T} \setminus \mathcal{T}_V} \deg(V)\deg(t) = \deg(V)\deg(\mathcal{T}).$$

We have obtain that for the case where V is an algebraic curve and $|\mathcal{S}| \ge k$ we can bound the number of incidences by

$$I(\mathcal{S}, \mathcal{T}) = I(\mathcal{S}, \mathcal{T}_V) + I(\mathcal{S}, \mathcal{T} \setminus \mathcal{T}_V) \le (b-1)|\mathcal{S}| + \deg(V)\deg(\mathcal{T}).$$

Notice that this bound is acceptable since for $d = 1$ we have $\alpha_k(1) = 0$, $\beta_k(1) = 1$ and $\tau_1(b, k) = k$ for all k and the theorem would have given us the bound

$$I(\mathcal{S}, \mathcal{T}) \lesssim_n 2k\deg(\mathcal{T})\deg(V) + (b-1)|\mathcal{S}|.$$

This bound is also bigger than $(k-1)\deg(\mathcal{T})$ therefore we are done with the base case.

Let us now suppose that $d > 1$ and assume that the result holds for every smaller dimension. Notice that if $|\mathcal{S}| < k$ we obtain the same bound as in the case of $d = 1$ so let us suppose $|\mathcal{S}| \ge k$. Consider the subset $\mathcal{T}_V \subseteq \mathcal{T}$ of hypersurfaces containing all of V. As it happened in the previous case, since $\mathcal{S}$ is (k, b)-free with respect to $\mathcal{T}$ we find that $|\mathcal{T}_V| = b'$ for some $0 \le b' \le b-1$. These elements contribute at most $b'|\mathcal{S}|$ incidences. Let us now consider the subset $\mathcal{T}' = \mathcal{T} \setminus \mathcal{T}_V$. Notice that every $t \in \mathcal{T}'$ satisfies $\dim(t \cap V) < d$. For every integer $0 \le s \le n-d$ we will consider the parameters

$$M_s = \left(\frac{b|\mathcal{S}|^k}{k^k \deg(\mathcal{T})(\prod_{i=1}^{s} \delta_i(V))^k} \right)^{\frac{1}{k(n-s)-1}} \tag{7.0.2}$$

Let us suppose that $M_{n-d} \lesssim_n 1$. Then it follows that

$$b|\mathcal{S}|^k \lesssim_n k^k \deg(\mathcal{T})(\prod_{i=1}^{n-d} \delta_i(V))^k \iff |\mathcal{S}| \lesssim_n kb^{-1/k}\deg(\mathcal{T})^{1/k} \prod_{i=1}^{n-d} \delta_i(V).$$

By Corollary 3.2.8 we conclude that

$$|\mathcal{S}| \lesssim_n kb^{-1/k}\deg(\mathcal{T})^{1/k}\deg(V).$$

From Lemma 7.0.2 we get the bound

$$\begin{aligned} I(\mathcal{S}, \mathcal{T}) &\lesssim_n b^{1/k}\left(kb^{-1/k}\deg(\mathcal{T})^{1/k}\deg(V) \right)|\mathcal{T}|^{1-1/k} + (k-1)|\mathcal{T}| \\ &\lesssim_n k\deg(\mathcal{T})^{1/k}\deg(V)|\mathcal{T}|^{1-1/k} + (k-1)|\mathcal{T}| \\ &\lesssim_n k\deg(\mathcal{T})\deg(V) + (k-1)|\mathcal{T}| \end{aligned}$$

where in the last inequality we have used that $\left(\frac{\deg(\mathcal{T})}{|\mathcal{T}|}\right)^{1/k}|\mathcal{T}| \leq \deg(\mathcal{T})$.

Let us suppose that $M_{n-d} \gtrsim_n 1$. We can now use the machinery of the polynomial method. Let $s = i_V(M_{n-d})$. Using Theorem 5.2.2 we find a real polynomial $P \notin I(V)$ with $\deg(P) \lesssim_n M_{n-d}$, such that each connected component of $\mathbb{R}^n \setminus Z(P)$ contains

$$\lesssim_n \frac{|\mathcal{S}|}{M_{n-d}^{n-s}\Delta_s(V)} \lesssim_n \frac{|\mathcal{S}|}{M_{n-d}^{n-s}\prod_{i=1}^{s}\delta_i(V)}$$

elements of $\mathcal{S}$.

Let V_{n-s} be an $(n-s)$-minimal variety of V. By definition we know that $\mathcal{S} \subseteq V \subseteq V_{n-s}$, $\dim(V_{n-s}) = n-s$, V_{n-s} is irreducible and $\deg(V_{n-s}) \sim_n \prod_{i=1}^{s}\delta_i(V)$. Write $\Omega_1, \ldots, \Omega_g$ for the connected components of $\mathbb{R}^n \setminus Z(P)$ and define the following sets:

$$\mathcal{S}_i := S \cap \Omega_i, \quad \mathcal{T}_i := \{t \in T' \mid (t \cap V_{n-s}) \cap \Omega_i \neq \emptyset\}.$$

Notice that the irreducible components of $t \cap V_{n-s}$ have dimension $n-s-1$ and by Lemma 3.1.5 their degrees sum up to at most $\deg(\mathcal{T})\deg(V_{n-s})$ which means that

$$\deg(t \cap V_{n-s}) \leq \deg(\mathcal{T})\deg(V_{n-s}).$$

Therefore by Theorem 1.0.15 we find that $(t \cap V_{n-s})(\mathbb{R})$ intersects

$$\lesssim_n \deg(t \cap V_{n-s})\deg(P)^{n-s-1} \leq \deg(\mathcal{T})\deg(V_{n-s})\deg(P)^{n-s-1}$$

connected components of $\mathbb{R}^n \setminus Z(P)$. More precisely, this means that each element $t \in \mathcal{T}'$ belongs to $\mathcal{T}_i$ for at most

$$\lesssim_n \deg(\mathcal{T})\deg(V_{n-s})\deg(P)^{n-s-1}$$

values of $1 \leq i \leq g$. To sum up, we have obtained the bounds

$$|\mathcal{S}_i| \lesssim_n \frac{|\mathcal{S}|}{M_{n-d}^{n-s}\prod_{i=1}^{s}\delta_i(V)} \sim_n \frac{|\mathcal{S}|}{M_{n-d}^{n-s}\deg(V_{n-s})},$$

$$|\mathcal{T}_i| \lesssim_n \deg(\mathcal{T})\deg(V_{n-s})\deg(P)^{n-s-1} \lesssim_n \deg(\mathcal{T})\deg(V_{n-s})M_{n-d}^{n-s-1}.$$

Using Lemma 7.0.2 we find the following bound for the number of incidences in each cell:

$$\begin{aligned}
&\leq \sum_{i=1}^{g} b^{1/k}|\mathcal{S}_i||\mathcal{T}_i|^{1-1/k} + (k-1)|\mathcal{T}_i| \\
&\lesssim_n b^{1/k}\frac{|\mathcal{S}|}{(M_{n-d}^{n-s}\deg(V_{n-s}))^{1-1/k}}(\deg(\mathcal{T})\deg(V_{n-s})M_{n-d}^{n-s-1})^{1-1/k} + k\deg(\mathcal{T})\deg(V_{n-s})M_{n-d}^{n-s-1} \\
&\lesssim_n b^{1/k}\frac{|\mathcal{S}|\deg(\mathcal{T})^{1-1/k}}{M_{n-d}^{1-1/k}} + k\deg(\mathcal{T})(\prod_{i=1}^{s}\delta_i(V))M_{n-d}^{n-s-1} \\
&= \frac{b^{1/k}|\mathcal{S}|\deg(\mathcal{T})^{1-1/k}M_{n-d}^{1/k}}{M_{n-d}} + \frac{k\deg(\mathcal{T})(\prod_{i=1}^{s}\delta_i(V))M_{n-d}^{n-s}}{M_{n-d}}.
\end{aligned} \tag{7.0.3}$$

Notice that the two summands in the previous bound are equal if and only if $M_s = M_{n-d}$. Indeed, we can rearrange (7.0.2) to obtain

$$(\prod_{i=1}^{s}\delta_i(V))^{-k}k^{-k} = M_s^{k(n-s)-1}\deg(\mathcal{T})b^{-1}|\mathcal{S}|^{-k} \iff (\prod_{i=1}^{s}\delta_i(V))k = M_s^{-(n-s)+1/k}\deg(\mathcal{T})^{-1/k}b^{1/k}|\mathcal{S}|. \tag{7.0.4}$$

If we substitute (7.0.4) into the numerator of the right hand-side summand of (7.0.3) we obtain:

$$k(\prod_{i=1}^{s}\delta_i(V))\deg(\mathcal{T})M_{n-d}^{n-s} = b^{1/k}|\mathcal{S}|\deg(\mathcal{T})^{1-1/k}M_{n-d}^{n-s}M_s^{-(n-s)+1/k}$$

which is equal to the numerator of the left hand-side summand if and only if $M_s = M_{n-d}$.

If we are able to show that $M_{n-d} \lesssim_n M_s$ then, since $1 \lesssim_n M_{n-d}$ the left hand-side summand in (7.0.3) dominates up to at most a $O_n(1)$-constant and therefore we will be able to obtain a bound for the number of incidences bounding this summand.

Claim 7.0.4. $M_{n-d} \lesssim_n M_s$.

Proof. Using that $M_{n-d} \lesssim_n \delta_{s+1}(V)$ we trivially get the following inequality:

$$\Big(\frac{b|\mathcal{S}|^k}{k^k\deg(\mathcal{T})(\prod_{i=1}^{n-d}\delta_i(V))^k}\Big) \lesssim_n \delta_{s+1}(V)^{dk-1}. \tag{7.0.5}$$

Doing some rearrangements and using (7.0.5) in the last inequality of the following computations we obtain the result:

$$\begin{aligned}
M_{n-d} &= \Big(\frac{b|\mathcal{S}|^k}{k^k\deg(\mathcal{T})(\prod_{i=1}^{s}\delta_i(V))^k}\Big)^{\frac{1}{k(n-s)-1}+\left(\frac{1}{dk-1}-\frac{1}{k(n-s)-1}\right)}\Big(\prod_{i=s+1}^{n-d}\delta_i(V)\Big)^{-\frac{k}{dk-1}} \\
&= M_s\Big(\frac{b|\mathcal{S}|^k}{k^k\deg(\mathcal{T})(\prod_{i=1}^{s}\delta_i(V))^k}\Big)^{\left(\frac{1}{dk-1}-\frac{1}{k(n-s)-1}\right)}\Big(\prod_{i=s+1}^{n-d}\delta_i(V)\Big)^{-\frac{k}{dk-1}} \\
&= M_s\Big(\frac{b|\mathcal{S}|^k}{k^k\deg(\mathcal{T})(\prod_{i=1}^{n-d}\delta_i(V))^k}\Big)^{\left(\frac{1}{dk-1}-\frac{1}{k(n-s)-1}\right)}\Big(\prod_{i=s+1}^{n-d}\delta_i(V)\Big)^{\frac{k}{dk-1}-\frac{k}{k(n-s)-1}}\Big(\prod_{i=s+1}^{n-d}\delta_i(V)\Big)^{-\frac{k}{dk-1}} \\
&= M_s\Big(\frac{b|\mathcal{S}|^k}{k^k\deg(\mathcal{T})(\prod_{i=1}^{n-d}\delta_i(V))^k}\Big)^{\left(\frac{1}{dk-1}-\frac{1}{k(n-s)-1}\right)}\Big(\prod_{i=s+1}^{n-d}\delta_i(V)\Big)^{-\frac{k}{k(n-s)-1}} \\
&= M_sM_{n-d}\Big(\frac{b|\mathcal{S}|^k}{k^k\deg(\mathcal{T})(\prod_{i=1}^{n-d}\delta_i(V))^k}\Big)^{\left(-\frac{1}{k(n-s)-1}\right)}\Big(\prod_{i=s+1}^{n-d}\delta_i(V)\Big)^{-\frac{k}{k(n-s)-1}} \\
&\lesssim_n M_s\delta_{s+1}(V)^{1-\frac{dk-1}{k(n-s)-1}-\frac{(n-d-s)k}{k(n-s)-1}} \lesssim_n M_s.
\end{aligned}$$

□

It remains to write down an explicit bound of the left hand-side summand of (7.0.3) in terms of $\tau_d(b,k), \alpha_k(d)$ and $b_k(d)$. Notice that we have

$$M_{n-d}^{1-1/k} = \frac{b^{\frac{k-1}{k(dk-1)}} |\mathcal{S}|^{\frac{k-1}{dk-1}}}{k^{\frac{k-1}{dk-1}} \deg(\mathcal{T})^{\frac{k-1}{k(dk-1)}} (\prod_{i=1}^{n-d} \delta_i(V))^{\frac{k-1}{dk-1}}}.$$

Plugging this into (7.0.3) we obtain

$$\frac{(b^{1/k}|\mathcal{S}| \deg(\mathcal{T})^{1-1/k})(k^{\frac{k-1}{dk-1}} \deg(\mathcal{T})^{\frac{k-1}{k(dk-1)}} (\prod_{i=1}^{n-d} \delta_i(V))^{\frac{k-1}{dk-1}})}{b^{\frac{k-1}{k(dk-1)}} |\mathcal{S}|^{\frac{k-1}{dk-1}}}. \tag{7.0.6}$$

Now, notice that the following equalities hold:

$$\beta_k(d) = \frac{d(k-1)}{dk-1} = 1 - 1/k + \frac{k-1}{k(dk-1)}, \quad 1-\beta_k(d) = \frac{d-1}{dk-1} = 1/k - \frac{k-1}{k(dk-1)},$$

$$\alpha_k(d) = \frac{k(d-1)}{dk-1} = 1 - \frac{k-1}{dk-1}$$

Substituting them into (7.0.6) we get

$$b^{1-\beta_k(d)} k^{1-\alpha_k(d)} |\mathcal{S}|^{\alpha_k(d)} \deg(\mathcal{T})^{\beta_k(d)} (\prod_{i=1}^{n-d} \delta_i(V))^{1-\alpha_k(d)} = \tau_d(b,k) |\mathcal{S}|^{\alpha_k(d)} \deg(\mathcal{T})^{\beta_k(d)} (\prod_{i=1}^{n-d} \delta_i(V))^{1-\alpha_k(d)}$$

Taking into account that $\deg(V) \leq \prod_{i=1}^{n-d} \delta_i(V)$ we finally get the bound

$$I(\mathcal{S}, \mathcal{T}') \lesssim_n \tau_d(b,k) |\mathcal{S}|^{\alpha_k(d)} \deg(\mathcal{T})^{\beta_k(d)} \deg(V)^{1-\alpha_k(d)}.$$

Let us summarize what we have done. We have found the bound $I(\mathcal{S}, \mathcal{T}_V) \lesssim_n b'|\mathcal{S}|$ for $1 \leq b' \leq (b-1)$. Moreover, if $M_{n-d} \lesssim_n 1$ then $I(\mathcal{S}, \mathcal{T}') \lesssim_n k \deg(\mathcal{T}) \deg(V)$. Summing both bounds gives us

$$I(\mathcal{S}, \mathcal{T}) \lesssim_n k \deg(\mathcal{T}) \deg(V) + b'|\mathcal{S}|. \tag{7.0.7}$$

On the other hand, if $M_{n-d} \gtrsim_n 1$ then

$$I(\mathcal{S}, \mathcal{T}) \lesssim_n \tau_d(b,k) |\mathcal{S}|^{\alpha_k(d)} \deg(\mathcal{T})^{\beta_k(d)} \deg(V)^{1-\alpha_k(d)} + b'|\mathcal{S}|. \tag{7.0.8}$$

Both (7.0.7) and (7.0.8) can be bounded by (1.0.18) and we are done counting incidences inside the cells defined by $Z(P)$.

To finish the proof we have to deal with the incidences coming from $Z(P) \cap V$. Let $W_1, \ldots, W_r$ be the irreducible components of $Z(P) \cap V$, that is, $Z(P) \cap V = W_1 \cup \cdots \cup W_r$ with $\dim(W_i) = d-1$ for all $1 \leq i \leq r$. Let us define the set

$$\mathcal{S}^{(i)} = \{s \in S \cap W_i \mid s \notin W_j,\ j < i\}$$

which is a partition of $\mathcal{S}$. Since each element of $\mathcal{T}_V$ contains $\mathcal{S}$ and $|\mathcal{T}_V| = b'$ we have that $\mathcal{S}$ is $(k, b-b')$-free with respect to $\mathcal{T}' = \mathcal{T} \setminus \mathcal{T}_V$. By induction hypothesis we obtain that

$$\begin{aligned} I(\mathcal{S}^{(i)}, \mathcal{T}') \lesssim_n & \; c_1 |S^{(i)}|^{\alpha_k(d-1)} \deg(T')^{\beta_k(d-1)} \deg(W_i)^{1-\alpha_k(d-1)} \\ & + k \deg(T') \deg(W_i) + (b + b' - 1)|S^{(i)}| \\ \lesssim_n & \tau_{d-1}(b,k) |S^{(i)}|^{\alpha_k(d-1)} \deg(\mathcal{T})^{\beta_k(d-1)} \deg(W_i)^{1-\alpha_k(d-1)} \\ & + k \deg(\mathcal{T}) \deg(W_i) + (b + b' - 1)|S^{(i)}|. \end{aligned}$$

Therefore, the number of incidences inside of $Z(P) \cap V$ is bounded by

$$\begin{aligned} \sum_{i=1}^{r} I(\mathcal{S}^{(i)}, \mathcal{T}') \lesssim_n & \tau_{d-1}(b,k) \deg(\mathcal{T})^{\beta_k(d-1)} \sum_{i=1}^{r} |S^{(i)}|^{\alpha_k(d-1)} \deg(W_i)^{1-\alpha_k(d-1)} \\ & + k \deg(\mathcal{T}) \sum_{i=1}^{r} \deg(W_i) + (b + b' - 1) \sum_{i=1}^{r} |S^{(i)}| \\ \lesssim_n & \tau_{d-1}(b,k) \deg(\mathcal{T})^{\beta_k(d-1)} |\mathcal{S}|^{\alpha_k(d-1)} (M_{n-d} \deg(V))^{1-\alpha_k(d-1)} \\ & + k M_{n-d} \deg(\mathcal{T}) \deg(V) + (b + b' - 1)|\mathcal{S}| \\ \lesssim_n & \tau_d(b,k) \deg(\mathcal{T})^{\beta_k(d)} |\mathcal{S}|^{\alpha_k(d)} \deg(V)^{1-\alpha_k(d)} \\ & + k \deg(\mathcal{T}) \deg(V) + (b + b' - 1)|\mathcal{S}|. \end{aligned}$$

Summing this last bound with the bounds obtained at (7.0.7) and (7.0.8) we get the result. □

We finish this chapter with an observation.

Taking $V = \mathbb{C}^n$, $\mathcal{T}$ a set of hypersurfaces with every element of degree $O_n(1)$ and $b = O(1)$ in Theorem 1.0.18 we get the following result.

Corollary 7.0.5. *Let $\mathcal{T}$ be a set of hypersurfaces of $\mathbb{R}^n$ of degree $O(1)$ and $\mathcal{S}$ a set of points that is $(k, O_n(1))$-free with respect to $\mathcal{T}$. Then*

$$I(\mathcal{S}, \mathcal{T}) \lesssim_{n,k} |\mathcal{S}|^{\alpha_k(n)} |\mathcal{T}|^{\beta_k(n)} + |\mathcal{T}| + |\mathcal{S}|.$$

Let $n = 2$ and $k = 2$. Since $\alpha_2(2) = \beta_2(2) = \frac{2}{3}$ we see that the above corollary recovers Theorem 2.2.9.

Chapter 8

Bounds for the number of connected components of a variety

The aim of this chapter is to prove Theorem 1.0.19. For this, we will build on the work of Barone and Basu [BB15].

Obtaining bounds for the number of connected components of real algebraic varieties is a problem that goes back to the seminal work of Milnor [Mil64] and Thom [Tho65] in the 1960s. Unfortunately, the bounds that they were able to obtain are not sufficient to get the best possible results when attacking a problem using the polynomial partitioning theorem. In particular, we need a result that give the explicit dependence on the degrees of the polynomials which define the varieties that are being studied. As we mentioned before, this was first studied by Barone and Basu in [BB15]. A simplified version (given in [Wal18]) of their main result is the following.

Theorem 8.0.1. *Let $f_1, \dots, f_{n-d} \in \mathbb{R}[x_1, \dots, x_n]$ be polynomials with $\deg(f_1) \leq \cdots \leq \deg(f_{n-d})$. Then if the real dimension of $Z(f_1, \dots, f_i)$ is at most $n - i$ for every $1 \leq i \leq n - d$, we have the bound*

$$b_0(Z(f_1, \dots, f_{n-d})(\mathbb{R})) \lesssim_n \left(\prod_{i=1}^{n-d} \deg(f_i)\right) \deg(f_{n-d})^d.$$

In order to obtain a similar result for algebraic varieties of arbitrary dimension we need to remove the restriction on the real dimension of $Z(f_1, \dots, f_i)$. The results that were studied in chapter 6 about envelopes and the ones that we will see in this chapter regarding *full covers* allow us to do just that.

Most of the material contained in this chapter is extracted from [Wal18, Section 7].

8.1 Full covers

The concept of *full cover* allows us to control the higher dimensional components produced by the admissible tuples of the irreducible components of an envelope. That is, we know that we can control the irreducible components of an envelope of an algebraic variety. We are now interested in obtaining control on the irreducible components of this envelope. To this end we will prove Lemma 8.1.2, Proposition 8.1.3 and Corollary 8.1.5 building on the work on envelopes done in chapter 6 and some of the results obtained in chapter 3.

Definition 8.1.1 (Full cover)**.** Let $V \subseteq \mathbb{C}^n$ be an irreducible algebraic variety. If $\dim(V) = n-1$, we say an algebraic set $\mathcal{F}(V)$ is a full cover of V if $\mathcal{F}(V) = V$. Recursively, let $\dim(V) = d < n-1$. We say an algebraic set $\mathcal{F}(V)$ is a K-full cover of V if there exists a K-admissible tuple of polynomials Q for V such that

$$\mathcal{F}(V) = \mathcal{S}_V^{n-d}(Q) \cup \bigcup_{W_i \subseteq \mathcal{E}_V(Q)} \mathcal{F}(W_i),$$

where the last union runs through all the irreducible components W_i of $\mathcal{E}_V(Q)$ and each $\mathcal{F}(W_i)$ is a K-full cover of W_i.

Lemma 8.1.2. *Let $\mathcal{F}(V)$ be a K-full cover of V and let $W_1, \ldots, W_r$ be the irreducible components of $\mathcal{F}(V)$. Then*

$$\sum_{i=1}^{r} \deg(W_i) \lesssim_{K,n} \deg(V).$$

Proof. We do the proof by induction on the codimension. Since $\mathcal{F}(V) = V$ if $\dim(V) = n-1$, the result is trivial in this case. Let $d = \dim(V) \leq n-2$ and assume that the result holds for every variety of larger dimension.
Let $T_1, \ldots, T_s$ be the irreducible components of $\mathcal{F}(V)$ coming from $\mathcal{S}_V^{n-d}(Q)$. By Lemma 6.1.7 we know that

$$\sum_{i=1}^{s} \deg(T_i) \lesssim_{K,n} \delta_1(V) \cdots \delta_{n-d}(V) \sim_n \deg(V),$$

so we are done in this case.
The other irreducible components are those of K-full covers $\mathcal{F}(W)$ where W is an irreducible component of $\mathcal{E}_V(Q)$. Let us fix an irreducible component W of $\mathcal{E}_V(Q)$. Observe that by definition of envelope it must be that $\dim(W) > \dim(V)$. Let $R_1, \ldots, R_t$ be the irreducible components of W. Then, by induction we know that

$$\sum_{i=1}^{t} \deg(R_i) \lesssim_{K,n} \deg(W).$$

Moreover, by Corollary 6.1.8 we know that the sum of the degrees of the irreducible components of $\mathcal{E}_V(Q)$ having dimension $n-k$ is bounded by

$$\lesssim_{K,n} \delta_1(V) \cdots \delta_k(V),$$

for some $1 \leq k \leq n-d$. Therefore if we sum all the degrees of all irreducible components of $\mathcal{E}_V(Q)$ we obtain the bound

$$\lesssim_{K,n} \delta_1(V) + \cdots + \prod_{k=1}^{n-d} \delta_k(V) \lesssim_{K,n} \prod_{k=1}^{n-d} \delta_k(V) \sim_n \deg(V),$$

this concludes the proof. □

Proposition 8.1.3. *Let $V \subseteq \mathbb{C}^n$ be an irreducible variety of dimension d and let $\epsilon > 0$ be given. Then V admits an $O_{\epsilon,n}(1)$-full cover $\mathcal{F}(V)$ such that, for every $1 \leq k \leq n-d-1$, every irreducible component of $\mathcal{F}(V)$ of dimension $n-k$ has degree at most $\epsilon\delta_1(V)\cdots\delta_k(V)$.*

Proof. We do induction on the codimension. The result is clear when $d = n-1$. Let $d = \dim(V) \leq n-2$ and suppose that the result holds for all varieties of larger codimension. Choose $\epsilon = \epsilon_0 > \epsilon_1 > \cdots > \epsilon_{n-d-1} > 0$ such that ϵ_i is sufficiently small with respect to n and ϵ_{i-1}. Let Q be a $(C_1, \ldots, C_{n-d})$-admissible tuple for V of the form given by Proposition 6.1.5 with respect to the parameters ϵ_i defined above. Recall that Proposition 6.1.5 guarantees that the union of all $(n-d)$-dimensional irreducible components of $\mathcal{E}_V(Q)$ has degree less than $\epsilon_d\delta_1(V)\cdots\delta_d(V)$. By Lemma 6.1.4 we know that the $(n-d)$-dimensional components of $\mathcal{E}_V(Q)$ are also irreducible components of $\mathcal{S}_V^d(Q)$. Therefore we can take these irreducible components as components of the full cover that we are constructing. The following claim gives us the remaining irreducible components.

Claim 8.1.4. *Let Q be the admissible tuple defined above. For every irreducible component W of $\mathcal{E}_V(Q)$ there is a full cover with all its irreducible components of dimension $n-k \geq \dim(W)$ of degree at most $\epsilon\delta_1(V)\cdots\delta_k(V)$.*

Proof. By Lemma 6.1.4 we know that an irreducible component W of $\mathcal{E}_V(Q)$ having dimension $\dim(W)$ is also an irreducible component of $\mathcal{S}_V^{n-\dim(W)}$, thus W is an irreducible component of $Z(Q_1, \ldots, Q_{n-\dim(W)})$.
Moreover, since $V \subseteq W$ we can use Corollary 3.2.13 to find that

$$\deg(W) \gtrsim_n \delta_1(V)\cdots\delta_{n-\dim(W)}(V).$$

By Lemma 6.1.3 we know that

$$\delta_i(W) \lesssim_{\epsilon_{i-1},n} \delta_i(V) \text{ for all } 1 \leq i \leq n-\dim(W) \tag{8.1.1}$$

Let $\epsilon_W \gtrsim_{n,\epsilon_{n-\dim(W)-1}} 1$ be a sufficiently small constant with respect to n and $n-\dim(W)-1$. Notice that by definition of envelope we have that $\dim(W) > \dim(V)$. Applying the induction hypothesis we find an $O_{\epsilon_W,n}(1)$-full cover $\mathcal{F}(W)$ of W such that, if $n-k > \dim(W)$, then all its irreducible components of dimension $n-k$ have degree at most $\epsilon_W\delta_1(W)\cdots\delta_k(W)$. Therefore applying the inequalities in (8.1.1) to this bound we get

$$\epsilon_W\delta_1(W)\cdots\delta_k(W) \lesssim_{\epsilon_{k-1},n} \epsilon_W\delta_1(V)\cdots\delta_k(V) \leq \epsilon\delta_1(V)\cdots\delta_k(V).$$

It remains to consider the irreducible components of $\mathcal{F}(W)$ which have dimension $\dim(W)$. By Lemma 6.1.7 and Lemma 6.1.3 we know that their degree is bounded by

$$\leq_{n,\epsilon_W} \delta_1(V)\cdots\delta_{n-\dim(W)}(V) \sim_n \deg(W).$$

Moreover, since W is an irreducible component of $\mathcal{E}_V(Q)$ and by the properties of Q we have that

$$\deg(W) \leq_{n,\epsilon_W} \epsilon_{n-\dim(W)}\delta_1(V)\cdots\delta_{n-\dim(W)}(V) \leq \epsilon\delta_1(V)\cdots\delta_{n-\dim(W)}(V).$$

□

This finishes the proof. □

Corollary 8.1.5. *Every irreducible variety $V \subseteq \mathbb{C}^n$ admits an $O_n(1)$-full cover $\mathcal{F}(V)$ such that V is an irreducible component of $\mathcal{F}(V)$.*

Proof. We proceed by contradiction. Let $\epsilon > 0$ be a sufficiently small constant with respec to n. Let $\mathcal{F}(V)$ be an $O_{\epsilon,n}(1)$-full cover of V of the form provided by Proposition 8.1.3. Suppose that $\mathcal{F}(V)$ has an irreducible component W of dimension $n-k > \dim(V)$ containing V. By Proposition 8.1.3 of this particular cover we know that

$$\deg(W) \leq \epsilon\delta_1(V)\cdots\delta_k(V).$$

This is a contradiction since Corollary 3.2.13 tells us that it must be

$$\deg(W) \gtrsim_n \delta_1(V)\cdots\delta_k(V).$$

Therefore the only possibility is that $V = W$ and we are done. □

8.2 Proof of the main theorem

The last tool that we need in order to prove Theorem 1.0.19 is the next result.

Theorem 8.2.1. *Let $V \subseteq \mathbb{C}^n$ be an irreducible variety of dimension d and let $\mathcal{F}(V)$ be a K-full cover of V. Then the number $b_0(\mathcal{F}(V)(\mathbb{R}))$ of connected components of $\mathcal{F}(V)(\mathbb{R})$ satisfies*

$$b_0(\mathcal{F}(V)(\mathbb{R})) \lesssim_{K,n} \deg(V)\delta(V)^d.$$

Proof. We do induction on the codimension. The result is known to be true when $d = n-1$ [Mil64] therefore we assume that $d < n-1$ and that the result holds for all varieties of dimension larger than d. Take $Q = \{Q_1, \ldots, Q_{n-d}\}$ to be the K-admissible tuple for V associated to $\mathcal{F}(V)$ and define the following algebraic set:

$$X := \mathcal{S}_V^{n-d}(Q) \setminus \bigcup_{W_i \subseteq \mathcal{E}_V(Q)} \mathcal{F}(W_i),$$

where the union goes through all the irreducible components of $\mathcal{E}_V(Q)$ and where $\mathcal{F}(W_i)$ is the K-full cover of W_i associated to $\mathcal{F}(V)$. We need the following observation.

Claim 8.2.2. *Let V and X be as above. Then every irreducible component of $\mathcal{F}(V)(\mathbb{R})$ intersecting X also intersects $Z_\tau(Q)$ for some $(n-d)$-tuple such that $\tau \leq (n-1, n-2, \ldots, d)$.*

Proof. If $x \in X(\mathbb{R})$ then by definition of $\mathcal{S}_V^{n-d}(Q)$ we know that $x \in Z_{n-d}(Q)$. We also know that $\mathcal{E}_V(Q) \subseteq \bigcup_{W_i \subseteq \mathcal{E}_V(Q)} \mathcal{F}(W_i)$ which means that $x \notin \mathcal{E}_V^{n-d}(Q)$. Using Lemma 6.2.6 we find that

$$\dim^{\mathbb{R}}_{Q,(n-d)}(x) \leq (n-1, n-2, \ldots, d).$$

Therefore, $x \in Z_\tau(Q)$ for some $(n-d)$-tuple $\tau \leq (n-1, n-2, \ldots, d)$ proving the claim. □

Using the claim and the fact that $Z_\tau(Q) \subseteq Z_{(n-d)}(Q) \subseteq \mathcal{F}(V)$ we get that the number of connected components of $\mathcal{F}(V)(\mathbb{R})$ that intersects X is at most the sum over all $\tau \leq (n-1, n-2, \ldots, d)$ of the number of connected components of $Z_{n-d}(Q)(\mathbb{R})$ that intersect $Z_\tau(Q)$. By Corollary 6.2.7 we know that this number is bounded by $\lesssim_{n,K} \delta(V)^d \deg(V)$.

To finish the proof we need to bound the number of connected components of $\mathcal{F}(V)$ which do not intersect X. Observe that all this connected components are contained inside of

$$\bigcup_{W_i \subseteq \mathcal{E}_V(Q)} \mathcal{F}(W_i),$$

therefore we only need to obtain a bound for the number of connected components of this last set.

By definition of envelope we know that $\dim(W_i) > \dim(V)$ for each irreducible component $W_i \subseteq \mathcal{E}_V(Q)$. Therefore, by induction we find that

$$b_0(\mathcal{F}(W_i)(\mathbb{R})) \lesssim_{K,n} \deg(W_i)\delta(W_i)^{\dim(W_i)}.$$

Thus, considering all the components we get

$$b_0\Big(\bigcup_{W_i \subseteq \mathcal{E}_V(Q)} \mathcal{F}(W_i)(\mathbb{R})\Big) \lesssim_{n,K} \sum_{W_i \subseteq \mathcal{E}_V(Q)} \deg(W_i)\delta(W_i)^{\dim(W_i)}. \tag{8.2.1}$$

Now take an irreducible component $W_i \subseteq \mathcal{E}_V(Q)$ such that $\dim(W_i) = n-k$ for some K. By Lemma 6.1.4 we know that $W_i \in \mathcal{S}_V^k(Q)$ which means that $W_i \subseteq Z_{n-k}(Q)$. By Lemma 6.1.3 we find that $\delta(W_i) \lesssim_{K,n} \delta_k(V)$. Moreover, by Corollary 6.1.8 we know that the sum of the degrees of the $(n-k)$-dimensional irreducible components of $\mathcal{E}_V(Q)$ is bounded by $\lesssim_{K,n} \delta_1(V) \cdots \delta_k(V)$. With this we get the bound

$$\sum_{\substack{W_i \subseteq \mathcal{E}_V(Q) \\ \dim(W_i) = n-k}} \delta(W_i)\delta(W_i)^{\dim(W_i)} \lesssim_{K,n} \delta_1(V) \cdots \delta_k(V)\delta_k(V)^{n-k} \lesssim_{K,n} \deg(V)\delta(V)^d.$$

Therefore we can conclude that the sum in (8.2.1) is bounded by

$$\lesssim_{K,n} \deg(V)\delta(V)^d$$

as we wanted to see. □

We have all the necessary tools to give a proof of the main theorem of this chapter.

Proof of Theorem 1.0.19. By Corollary 8.1.5 we know that V admits a $O_n(1)$-full cover $\mathcal{F}(V)$ such that V is an irreducible component of $\mathcal{F}(V)$. It suffices to take

$$X = \mathcal{F}(V).$$

Indeed, by Lemma 8.1.2 we have the bound

$$\deg(\mathcal{F}(V)) \lesssim_n \deg(V).$$

Finally, by Theorem 8.2.1 we obtain

$$b_0(\mathcal{F}(V)(\mathbb{R})) \lesssim_n \deg(V)\delta(V)^d,$$

as desired. □

Bibliography

[BB15] Sal Barone and Saugata Basu. *On a real analog of Bezout inequality and the number of connected components of sign conditions.* Proceedings of the London Mathematical Society, 112(1):115–145, 2015. 41, 46, 59

[Bou91] J. Bourgain. *Besicovitch Type Maximal Operators and Applications to Fourier Analysis.* Geometric and functional analysis, 1(2):147–187, 1991. 2

[BS16] Saugata Basu and Martín Sombra. *Polynomial Partitioning on Varieties of Codimension Two and Point-Hypersurface Incidences in Four Dimensions.* Discrete and Computational Geometry, 55(1):158–184, 2016. 5

[CEG+91] Bernard Chazelle, Herbert Edelsbrunner, Leonidas J. Guibas, Richard Pollack, Raimund Seidel, Micha Sharir, and Jack Snoeyink. *Counting and cutting cycles of lines and rods in space.* IEEE Transactions on Industry Applications, 27(1 pt 1):242–251, 1991. 3

[Dav71] Roy O. Davies. *Some remarks on the kakeya problem.* Mathematical Proceedings of the Cambridge Philosophical Society, 69(3):417–421, 1971. 2

[DGL17] Xiumin Du, Larry Guth, and Xiaochun Li. *A sharp Schrödinger maximal estimate in* $\mathbb{R}^2$. Annals of Mathematics, 186(2):607–640, 2017. 4

[Dvi08] Zeev Dvir. *On the size of Kakeya sets in finite fields.* Journal of the American Mathematical Society, 22(4):1093–1097, 2008. 1, 2

[Dvi10] Zeev Dvir. *Incidence theorems and their applications.* Foundations and Trends in Theoretical Computer Science, 6(4):257–293, 2010. 1

[Erd70] Paul Erdos. *On Sets of Distances of n Points.* The American Mathematical Monthly, 77(7):738, 1970. 1, 3

[ES11] György Elekes and Micha Sharir. *Incidences in three dimensions and distinct distances in the plane.* In *Combinatorics Probability and Computing*, volume 20 of *SoCG '10*, pages 571–608, New York, NY, USA, 2011. ACM. 4

[FPS+14] Jacob Fox, János Pach, Adam Sheffer, Andrew Suk, and Joshua Zahl. *A semi-algebraic version of Zarankiewicz's problem*, 2014, arXiv:1407.5705v1. 5

[GK10] Larry Guth and Nets Hawk Katz. *Algebraic methods in discrete analogs of the Kakeya problem.* Advances in Mathematics, 225(5):2828–2839, 2010. 3

[GK15] Larry Guth and Nets Hawk Katz. *On the Erdos distinct distances problem in the plane.* Annals of Mathematics, 181(1):155–190, 2015. 1, 3, 4

[Gut14] Larry Guth. *A restriction estimate using polynomial partitioning.* arXiv e-prints, page arXiv:1407.1916, Jul 2014, 1407.1916. 4

[Gut15] Larry Guth. *Polynomial partitioning for a set of varieties.* Mathematical Proceedings of the Cambridge Philosophical Society, 159(3):459–469, 2015. 5

[Gut16] Larry Guth. *Polynomial methods in combinatorics.* University Lecture Series. American Mathematical Society, 2016. 1, 9

[Gut18] Larry Guth. *Restriction estimates using polynomial partitioning II.* In *Acta Mathematica*, volume 221, pages 81–142, 2018. 4

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. 20

[KMS12] Haim Kaplan, J. Matoušek, and Micha Sharir. *Simple Proofs of Classical Theorems in Discrete Geometry via the Guth-Katz Polynomial Partitioning Technique.* Discrete and Computational Geometry, 48(3):499–517, 2012. 9

[KT02] Nets Hawk Katz and Terence Tao. *New bounds for Kakeya problems.* Journal d'Analyse Mathematique, 87:231–263, 2002. 2

[Mil64] J. Milnor. *On the Betti Numbers of Real Varieties.* Proceedings of the American Mathematical Society, 15(2):275, 1964. 7, 47, 59, 62

[MP15] Jiří Matoušek and Zuzana Patáková. *Multilevel Polynomial Partitions and Simplified Range Searching.* Discrete and Computational Geometry, 54(1):22–41, 2015. 5

[OW17] Yumeng Ou and Hong Wang. *A cone restriction estimate using polynomial partitioning.* arXiv e-prints, page arXiv:1704.05485, Apr 2017, 1704.05485. 4

[ST42] A. H. Stone and J. W. Tukey. *Generalized "sandwich" theorems.* Duke Mathematical Journal, 9(2):356–359, 1942. 14

[ST83] E. Szemerédi and W. T. Trotter. *Extremal problems in discrete geometry.* Combinatorica, 3(3-4):381–392, 1983. 4, 14

[ST12] József Solymosi and Terence Tao. *An Incidence Theorem in Higher Dimensions.* Discrete and Computational Geometry, 48(2):255–280, 2012. 5

[Szé97] László A. Székely. *Crossing Numbers and Hard Erdos Problems in Discrete Geometry.* Combinatorics Probability and Computing, 6(3):353–358, 1997. 14

[Tao14] Terence Tao. *Algebraic combinatorial geometry: the polynomial method in arithmetic combinatorics, incidence combinatorics, and number theory.* EMS Surveys in Mathematical Sciences, 1(1):1–46, 2014. 1

[Tho65] Ronald M Thom. *Sur l"homologie des vari'et'es alg'ebriques r'eelles.* In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*, pages 255–265. Princeton Univ. Press, 1965. 7, 47, 59

[Wal18] Miguel N. Walsh. *Polynomial partitioning over varieties.* 2018, arXiv:1811.07865. 1, 5, 19, 29, 35, 59

[Wol96] Thomas Wolff. *Recent work connected with the Kakeya problem.* Prospects in mathematics, pages 129–162, 1996. 2

[WYZ13] Hong Wang, Ben Yang, and Ruixiang Zhang. *Bounds of incidences between points and algebraic curves*, 2013, arXiv:1308.0861. 4

[Zah15] Joshua Zahl. *A Szemerédi–Trotter Type Theorem in* $\mathbb{R}^4$. Discrete and Computational Geometry, 54(3):513–572, 2015. 5